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Some Topics on Limiters and FM
Demodulators

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Maung Gyi

July 1965

Technical Report No. 7050-3

Prepared under

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SOME TOPICS ON LIMITERS AND FM DEMODULATORS

by

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Systems Theory Laboratory
Stanford Electronics Laboratories
Stanford University Stanford, California

ABSTRACT

An analysis is made of wide-deviation frequency-modulation systems having low nominal-carrier-frequency to information-bandwidth ratios. Since limiting plays an important role in such systems, the effects of hard limiting of many signals in random noise are analyzed. Expressions are given for the output signal-to-signal ratios (SSR), the output signal-to-noise ratios (SNR), and the power in any crossproduct. The effect of the power in each output component as a function of the input SNR is investigated. It is found that for all values of input SNR's greater than 10, the strengths of the various output components are relatively constant. For the case of more than two input signals, a weak signal-boosting effect manifests itself when the input SNR's are less than 1 and the input SSR's are greater than $1/10$. The signal-suppression effect and the signal-to-signal power sharing, together with their dependence on input signal noise, are presented for various cases.

Expressions are presented for the autocorrelation function and the spectral power density of hard-limited FM pulse trains, which allow the computation of the intermodulation products in the information bandwidth or in any other frequency interval. The effect of the nominal-carrier frequency on the interference ratio is exhibited graphically for various constant deviations.

A partial interaction of the positive and negative spectra of the modulated wave causes extraneous outputs. For all such spurious components to be filterable, a relation is derived between the carrier frequency, the maximum frequency of information, and the spectral bandwidth of the modulated wave. The analysis results in systems which have better performance capabilities.

CONTENTS

	<u>Page</u>
I. INTRODUCTION	1
A. Background	1
B. Outline of Presentation	2
II. HARD LIMITING OF p SIGNALS IN RANDOM NOISE	4
A. Formulation of the Problem	4
1. Limiter Input Random Process	4
2. Input Noise	5
3. Autocorrelation Function of the Input Noise	5
4. Limiter Transfer Function	6
5. Narrowband Assumption	6
B. Limiter Output	6
1. Derivation of Autocorrelation Function	6
2. Spectral Analysis and Filtering	20
3. Output Signal-to-Noise Terms	22
C. Output Factors Due to Signal and Noise Interaction	27
D. Smooth Limiters with Error-Function Transfer Characteristics	29
E. Asymptotic Results	
1. Case I, Weak Signals	29
2. Case II, One Strong Signal	34
3. Case III, p Signals of Equal Strength	40
F. Computed Results	49
G. Conclusions to Chapter II	50
III. OUTPUT SPECTRA FOR WIDEBAND FM PULSE TRAINS GENERATED FROM HARD-LIMITED FM SIGNALS	68
A. Introduction	68
B. Input Process	69
1. Assumption	69
2. FM Pulse Train	71
C. Evaluation of $[y(t) y(t - T)]$	74
1. Expression of the 0 th Spectral Zone, C_0	74
2. Expression for the 2 ^p th Spectral Zone, $p \neq 0$	75

	<u>Page</u>
D. Output Process	80
1. Expansion of $z(t)$	80
2. Autocorrelation Function	81
3. Power Spectral Density	82
4. Output Voltage Vs Frequency Characteristic	85
E. Intermodulation Products	85
1. Interference Ratio, $IR(p,n)$	86
2. Pulse Shape and Interference Ratio	88
F. Results	88
1. Theoretical and Experimental Results	88
2. Advantages over Conventional Detectors	91
IV. REDUCTION OF SPURIOUS OUTPUTS IN LOW-CARRIER, WIDE- DEVIATION, WIDE-BANDWIDTH FM SYSTEMS	93
A. Formulation of the Problem	93
B. Analysis of the Problem	95
C. Interrelationships of f_c , f_{ym} , and f_{lp} for Filterable Spurious Outputs	97
D. Reduction of Spurious Outputs	98
1. Filtering Technique	99
2. Phasing Technique	99
V. CONCLUDING DISCUSSION	105
APPENDIX A. Useful Integrals of Bessel Functions	107
APPENDIX B. Derivation of Autocorrelation Function of the Output $z(t)$	110
REFERENCES	122

TABLES

<u>Number</u>	<u>Page</u>
1 Suppression effect for the three-signal case ($p = 3$) . . .	57
2 Suppression effect for the four-signal case ($p = 4$) . . .	58
3 Output signal power for p equal signals and no noise . .	61
4 Output signal power for p equal signals in noise	62
5 Crossproducts vs (S/N) for the four-signal case	63
6 Crossproducts for large p in noise (equal-signal case) .	64
7 Crossproducts for large p , no noise (equal-signal case) .	65
8 Interference ratios for various deviations and bandwidths .	89
9 Comparison of computed and measured values for various IR(p,n)	91

ILLUSTRATIONS

Figure

1 Block diagram of the receiver analyzed	4
2 Contours of integration	8
3 Output signal power for the three-signal case	54
4 Output signal power for the four-signal case	55
5 Output signal power for the five-signal case	56
6 Suppression effect for $p = 3$	59
7 Suppression effect for $p = 4$	60
8 Crossproduct S_{10}^{3p-3} vs p for equal-signals-in-noise case	66
9 Crossproducts vs p for equal-signals, no-noise case . . .	67
10 Significant components of the fundamental and third- harmonic zones of the limiter output	70
11 Block diagram of the system analyzed	71
12 Curves showing the interference ratio vs deviation for fixed $f_c/f_{mm} = 2$	90
13 Curves showing spurious output behavior vs carrier frequency at constant deviation	90

<u>Figure</u>	<u>Page</u>
14 Typical bandwidth of channel, using vestigial frequency modulations	93
15 Modulated spectrum when $f_h \gg f_{ymm}$	94
16 Modulated spectrum when $f_h \approx f_{ymm}$	96
17 Modulated spectrum when $f_c < f_{ymm}$ (shows aliasing) . . .	97
18 Lowpass filtered output	98
19 Block diagram and spectra of the proposed system using the filtering technique	100
20 Demodulation process using the filtering technique	100
21 Block diagrams of the phasing technique	101

PRINCIPAL SYMBOLS

$b_{k,m_1 m_2 \dots m_p}$	$= (N/2)^{k/2} h_{k,m_1 m_2 \dots m_f}$ [see Eq. (2.29)]
C_{-f_c}	$\mathfrak{F}\{\cos \psi(t)\} * \delta(f + f_c)$
C_{+f_c}	$\mathfrak{F}\{\cos \psi(t)\} * \delta(f - f_c)$
$\left[C_{+f_c} \right]_+$	$C_{+f_c} \cdot \frac{1}{2} [1 + \operatorname{sgn}(f)]$
$\left[C_{+f_c} \right]_-$	$C_{+f_c} \cdot \frac{1}{2} [1 - \operatorname{sgn}(f)]$
$E(x)$	complete elliptic integral of the second kind $= E\left(x, \frac{\pi}{2}\right) = \int_0^{\pi/2} [1 - x^2 \sin^2 \phi]^{1/2} d\phi$
f_c	nominal-carrier frequency
f_m	modulating frequency
f_{mm}	maximum frequency present in the information
$\mathfrak{F}\{G\}$	Fourier transform of $[G]$
$g(t) \overset{k}{*} g(t)$	k self-convolutions of $g(t)$
$I_m(x)$	modified Bessel function of the first kind of order m (imaginary argument)
$IM(p,n)$	intermodulation product at frequency $(2pf_c + nf_m)$
$IR(p,n)$	interference ratio at frequency $(2pf_c + nf_m)$
$J_m(x)$	Bessel function of the first kind of order m (real argument)
k	percent deviation

$M_{\mathbf{x}}(w_t, w_t')$	two-dimensional joint characteristic function
$p(y)$	probability distribution function of (y)
$R_{\mathbf{x}}[t_1, t_2]$	$E[x(t_1) \cdot x(t_2)]$ = autocorrelation function of the real random process $x(t)$
$R_{\mathbf{x}}[0, \tau]$	$R_{\mathbf{x}}(\tau)$
$S(f)$	spectral power density = Fourier transform of $R(\tau)$ $= \int_{-\infty}^{\infty} R(\tau) e^{-i\omega\tau} d\tau, \quad \omega = 2\pi f$
$S_+(f)$	$S(f) \cdot \frac{1}{2} [1 + \operatorname{sgn}(f)]$
$S_-(f)$	$S(f) \cdot \frac{1}{2} [1 - \operatorname{sgn}(f)]$
S_{+f_c}	$\mathcal{U}\{\sin \psi(t)\} * \delta(f - f_c)$
$\hat{x}(t)$	Hilbert transform of $x(t)$ = Cauchy's principal value of $\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{x(s)}{t - s} ds$
β	modulation index $\triangleq \pm \frac{\Delta f}{f_m} = \left(\pm \frac{k}{100} \cdot \frac{f_c}{f_m} \right)$
$\delta(t)$	Dirac delta function = unit impulse at $t = 0$
c_m	Neumann numbers = $\begin{cases} 1 & \text{if } m = 0 \\ 2 & \text{if } m > 0 \end{cases}$
$\xi_{\tau}(t)$	random process defined by $[\psi(t) - \psi(t - \tau)]$
$\rho(\tau)$	$\frac{R(\tau)}{R(0)}$ = normalized autocorrelation function $= \int_{-\infty}^{\infty} G(f) e^{i\omega\tau} df, \quad \omega = 2\pi f$

$\sigma(f)$

Fourier transform of $\rho(\tau) = \int_{-\infty}^{\infty} \rho(\tau) e^{-i\omega\tau} d\tau$

$\phi_y(f)$

characteristic function which is equal to

$$E[e^{i\omega y}] = \int_{-\infty}^{\infty} e^{i\omega y} p(y) dy, \quad \omega = 2\pi f$$

$\psi(t)$

information in the case of phase modulation

$\dot{\psi}(t)$

information in the case of frequency modulation

*

convolution; that is,

$$g(t) * f(t) = \int_{-\infty}^{\infty} g(\alpha) f(t - \alpha) d\alpha = \int_{-\infty}^{\infty} f(\alpha) g(t - \alpha) d\alpha$$

$\text{sgn}(f)$

$$\begin{cases} 0 & \text{for } f < 0 \\ \frac{1}{2} & \text{for } f = 0 \\ 1 & \text{for } f > 0 \end{cases}$$

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I. INTRODUCTION

The purpose of this research is to analyze some of the problems peculiar to wide-deviation frequency-modulation systems having low nominal-carrier frequency to information-bandwidth ratios. Some form of frequency modulation is employed in many tape-recorder systems due to the fact that the effect of random envelope variations occurring during playback can be eliminated at the receiver by hard limiting prior to demodulation. A typical electromagnetic recording channel has a bandwidth from an upper limit of a few megacycles down to a lower limit of a few hundred cycles. If, for example, it is desired to record information down to dc, some form of modulation must be used before the information can be so recorded. The parameters of a typical channel are a low nominal-carrier frequency and the widest possible deviation ratio. The high deviation is needed to obtain an acceptable signal-to-noise ratio (SNR). Since the channel bandwidth is fixed, most of the modulation schemes result in reduction of the information bandwidth that can be recorded.

In order to maximize the information bandwidth, a vestigial technique is sometimes employed. At other times conventional frequency modulation is used. In either case, spurious outputs arise because the bandwidth of the channel is not wide enough to record the entire portion of the frequency-modulated spectrum that carries significant amounts of energy. The ultimate performance of such a system is often limited by these spurious outputs rather than by the SNR. In other instances, many signals are multiplexed on the same channel. Then it is of importance to be able to compute the strengths of various crossproducts and the signal-to-noise ratios of each signal.

A. BACKGROUND

The effects of hard limiting on signals and noise have been studied by various investigators from time to time. Using the techniques developed by Bennett and Rice [Ref. 1] and by Middleton [Refs. 2,3], Davenport studied the case of a single sine wave and additive gaussian noise when passed through a v^{th} law device and gave explicit results for the changes

in signal-to-noise ratio produced by an ideal limiter [Ref. 4]. Later Granlund [Ref. 5] and then Baghdady [Ref. 6] studied the problem of hard limiting of two sine waves without noise and their results included some expressions for the evaluation of the strengths of output signals and intermodulation products. Davenport and Root [Ref. 7] studied the problem of bandpass limiting of two sine waves in the presence of additive gaussian noise. Their investigations were reinforced recently by Jones [Ref. 8], who studied the problem in great detail in order to examine the effects of limiters in long pulse radars and other cases where substantial pulse overlaps occurred. Jones studied the case of two signals in noise and gave relations among the output signals, noise, and intermodulation products due to bandpass hard limiting in that case only.

The present study extends the analysis to the case of bandpass hard limiting of p signals in additive gaussian noise. It is, in short, a generalization of the problem investigated by Jones, using the nonlinear transform technique which is convenient and mathematically rigorous.[†] The results are shown to agree with those of (1) Jones, when $p = 2$; (2) Granlund and Baghdady, when $p = 2$ and when the input signal-to-noise ratio is made arbitrarily large (no-noise case); and (3) Davenport, when $p = 1$.

B. OUTLINE OF PRESENTATION

Chapter II presents an analysis of the effect of hard limiting on p signals in noise. Expressions are given for the spectral power density and the autocorrelation of the limited output, the output SNR's, the output signal-to-signal ratios (SSR), the signal-suppression effect, and the intensity of any crossproduct. Some curves and numerical results are also presented, and some asymptotic values are derived.

Chapter III presents a derivation of expressions giving the output spectrum of wideband frequency-modulated pulse trains generated from

[†] Prior to publication of this study, P. D. Shaft, who has also investigated the problem, presented a paper at the IEEE Convention in New York on 24 March 1965 [Ref. 9]. Mr. Shaft was formerly with Western Development Laboratories, Philco Corporation, and is now with the Stanford Research Institute, Menlo Park, California.

hard-limited frequency-modulated signals. Expressions are given for the spurious outputs in the demodulated signal. A system of indexing the spurious components is developed, and curves are given which show the dependence of spurious outputs on carrier frequency, deviation, and the information frequency. Interference ratios are defined, then computed and tabulated together with experimentally measured values.

Chapter IV presents an analysis of the "aliasing" phenomenon which is due to the interaction of the positive and the negative spectra when the frequency of the nominal frequency-modulated carrier has a value lower than half the spectral bandwidth of the modulated wave. Expressions are given that relate the nominal-carrier frequency to the maximum frequency of information. Analysis and block diagrams of proposed systems are presented which are improvements over existing systems.

II. HARD LIMITING OF p SIGNALS IN RANDOM NOISE

A. FORMULATION OF THE PROBLEM

The effects of hard limiting of p signals in random noise are analyzed, assuming the input conditions described in the following five subsections. The system studied is shown in the block diagram of Fig. 1, in which it is assumed that the input signals and noise are bandlimited by bandpass filtering centered around f_c .

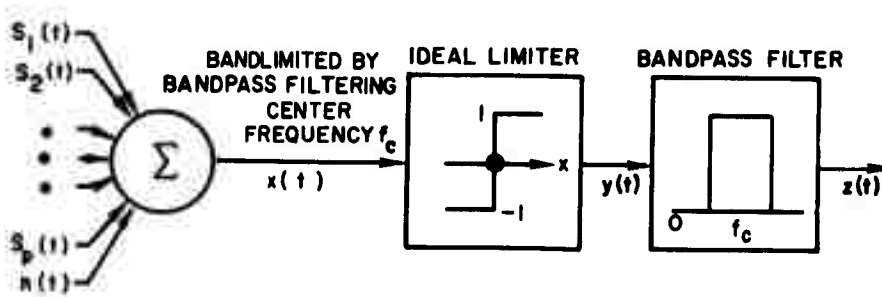


FIG. 1. BLOCK DIAGRAM OF THE RECEIVER ANALYZED.

1. Limiter Input Random Process

The limiter input random process, defined as $x(t)$, consists of p unrelated sine waves and additive gaussian noise. In particular, for every i in the $(1, p)$ interval,

$$s_i(t) = \sqrt{2S_i} \cos(\omega_i t + \phi_i) \quad (2.1)$$

where the ϕ_i are statistically independent random variables, each having a uniform distribution on the interval $(0, 2\pi)$.

The input $x(t)$ is therefore

$$\begin{aligned} x(t) &= s_1(t) + s_2(t) + \dots + s_p(t) + n(t) \\ &= \sum_{i=1}^p \sqrt{2S_i} \cos(\omega_i t + \phi_i) + n(t) \end{aligned} \quad (2.2)$$

2. Input Noise

The input noise $n(t)$ is a sample function of zero-mean, stationary, narrowband gaussian noise. It is assumed symmetrical around f_c since the signals and noise are bandpass filtered prior to limiting.

3. Autocorrelation Function of the Input Noise

The noise $n(t)$ is a sample function of a stationary random process. The covariance function of $n(t)$ can be readily written as [Davenport and Root, Ref. 7, p. 169]

$$\begin{aligned} R(t) &= R_c(t) \cos \omega_c t - R_{cs}(t) \sin \omega_c t \\ &= \left[R_c^2(t) + R_{cs}^2(t) \right]^{1/2} \left[\cos \omega_c t + \tan^{-1} \frac{R_{cs}(t)}{R_c(t)} \right] \end{aligned} \quad (2.3)$$

where $\omega_c = 2\pi f_c$.

Since f_c is chosen such that $S(f)$ is symmetric about it,

$$S_+(f + f_c) = S_+(-f + f_c) \quad (2.4)$$

and

$$S_-(f - f_c) = S_-(-f - f_c) \quad (2.5)$$

which gives

$$\left. \begin{aligned} S_c(f) &= 2S_+(f + f_c) \\ S_{cs}(f) &= 0 \end{aligned} \right\} \quad (2.6)$$

This equation implies that

$$R(t) = 2R_c(t) \cos \omega_c t \quad (2.7)$$

Adopting the usual normalization by defining $\rho(\tau)$ such that it contains unit power, one can write the covariance function of the input noise as

$$R_n(\tau) = N\rho(\tau) \cos \omega_c \tau \quad (2.8)$$

4. Limiter Transfer Function

The limiter output $y(t)$ can be written in terms of its input as

$$y(t) = g[x(t)] = \begin{cases} +1 & \text{for } x(t) > 0 \\ 0 & \text{for } x(t) = 0 \\ -1 & \text{for } x(t) < 0 \end{cases} \quad (2.9)$$

5. Narrowband Assumption

The narrowband assumption implies that the difference between any pair $[\omega_i, \omega_j]$ chosen from the set $\{\omega_1, \omega_2, \dots, \omega_p, \omega_c\}$ is negligible magnitudewise when compared to any member in the set; that is,

$$|\omega_i - \omega_j| \ll \omega_k \quad \text{for any } i, j \in \{k\} \quad (2.10)$$

where $\{k\} = \{1, 2, \dots, p, c\}$. This assumption guarantees that the interference between adjacent spectral zones at the output of the limiter will be negligible. Hence the first spectral zone can be obtained by suitable bandpass filtering around f_c .

B. LIMITER OUTPUT

1. Derivation of Autocorrelation Function

The signal-to-noise ratio among the output components of the limiter output $y(t)$ can be obtained from the autocorrelation function of the limiter output which is defined as

$$R_y(\tau) \triangleq E_x\{g[x(t)] g[x(t + \tau)]\} \quad (2.11)$$

or

$$R_y(t, t') = E_x\{g[x(t)] g[x(t')]\} \quad (2.12)$$

where $t' \triangleq t + \tau$. Using the techniques suggested by Bennett and Rice [Ref. 1], Middleton [Ref. 2], and Davenport and Root [Ref. 7], one can represent the transfer function of the nonlinear device by

$$g(x) = \frac{1}{2\pi j} \int_{C_+} f_+(w) e^{xw} dw + \frac{1}{2\pi j} \int_{C_-} f_-(w) e^{xw} dw \quad (2.13)$$

where

$$C_+ = \text{the contour } w = u' + jv, \quad u' > u_2$$

$$C_- = \text{the contour } w = u'' + jv, \quad u'' < u_3$$

$$f_+(w) = \int_0^\infty g_+(x) e^{-wx} dx = \int_0^\infty 1 \cdot e^{-wx} dx = \frac{1}{w}$$

$$f_-(w) = \int_{-\infty}^0 g_-(x) e^{-wx} dx = \int_0^\infty e^{-wx'} dx' = \frac{1}{w}$$

Hence

$$g(x) = \frac{1}{2\pi j} \int_{C_+} e^{xw} \frac{dw}{w} + \frac{1}{2\pi j} \int_{C_-} e^{xw} \frac{dw}{w}$$

The contours C_+ and C_- are shown in Fig. 2.

It should be noted that both contours include the entire $w = jv$ axis except the origin, which is excluded because of the presence of a

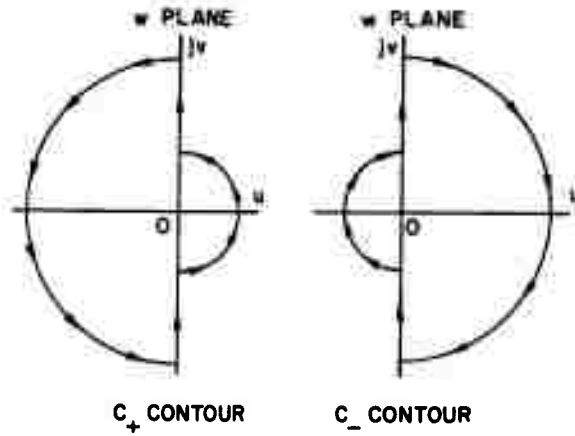


FIG. 2. CONTOURS OF INTEGRATION.

pole at that point. The integrands are identical and therefore, for compactness, the equation of $g(x)$ is represented by a single integral over the contour C . For final evaluation, one can simply evaluate over C_+ and C_- separately and add.

$$g(x) = \frac{1}{2\pi j} \int_C e^{wx} \frac{dw}{w} \quad (2.14)$$

Therefore,

$$R_y(t, t') = \frac{1}{(2\pi j)^2} \int_C \frac{dw_t}{w_t} \int_C \frac{dw_{t'}}{w_{t'}} E_x \{ \exp [w_t x(t) + w_{t'} x(t')] \} \quad (2.15)$$

By definition $E_x \{ \exp [w_t x(t) + w_{t'} x(t')] \}$ is the two-dimensional joint characteristic function and can be denoted by $M_x(w_t, w_{t'})$. Hence

$$R_y(t, t') = \frac{1}{(2\pi j)^2} \int_C \frac{dw_t}{w_t} \int_C \frac{dw_{t'}}{w_{t'}} M_x(w_t, w_{t'}) \quad (2.16)$$

Since the input process has an independent nature, manipulation of M_x is particularly convenient. The input $x(t)$ is the sum of the zero-mean, statistically independent random processes,

$$x(t) = n(t) + \sum_{i=1}^p s_i(t) \quad (2.17)$$

and therefore

$$\begin{aligned} M_x(w_t, w_{t'}) &= \prod_{i=1}^{p+1} M_{x_i}(w_t, w_{t'}) \\ &= \prod_{i=1}^{p+1} E_{x_i} \{ \exp [w_t x_i(t) + w_{t'} x_i(t')] \} \end{aligned} \quad (2.18)$$

It can be shown that the joint characteristic function for stationary gaussian noise is [Ref. 7, Eq. (13.43), p. 289; Parzen, Ref. 10]

$$M_n(w_t, w_{t'}) = \exp \left[\frac{N}{2} (w_t^2 + w_{t'}^2) + R_n(\tau) w_t w_{t'} \right] \quad (2.19)$$

Each of the sine-wave signals

$$s_i = \sqrt{2S_i} \cos (\omega_i t + \phi_i)$$

has a joint characteristic function

$$M_{s_i}(w_t, w_{t'}) = E [\exp (w_t \sqrt{2S_i} \cos \theta_t + w_{t'} \sqrt{2S_i} \cos \theta_{t'})] \quad (2.20a)$$

The exponentials are expanded using the Jacobi-Anger formula [Watson, Ref. 11; also see Eqs. (B.10a) - (B.10d) on p. 115, this report] to give

$$M_{s_i}(w_t, w_{t'}) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \epsilon_m \epsilon_n I_m(w_t \sqrt{2S_i}) I_n(w_{t'} \sqrt{2S_i}) E(\cos m\theta_t \cos n\theta_{t'}) \quad (2.20b)$$

The expectation can be simplified to

$$\begin{aligned}
 E(\cos m\theta_t \cos n\theta_{t'}) &= E[\cos m(\omega_i t + \phi_i) \cos n(\omega_i t' + \phi_i)] \\
 &= E\left\{\frac{1}{2} \cos[(m+n)\omega_i t + n\omega_i \tau + m\phi_i + n\phi_i]\right\} \\
 &\quad + E\left\{\frac{1}{2} \cos[(m-n)\omega_i t - n\omega_i \tau + (m-n)\phi_i]\right\} \\
 &= \begin{cases} 0 & \text{when } n \neq m \\ \frac{1}{2} \cos(n\omega_i \tau) & \text{when } m = n \end{cases} \quad (2.21)
 \end{aligned}$$

The above simplification gives

$$M_{s_i}(w_t, w_{t'}) = \sum_{m=0}^{\infty} \epsilon_m I_m(w_t \sqrt{2S_i}) I_m(w_{t'} \sqrt{2S_i}) \cos(n\omega_i \tau) \quad (2.22)$$

where

$I_m(x)$ = [modified Bessel function of the first kind of order m]

$$\epsilon_m = \text{Neumann numbers} = \begin{cases} 1 & \text{if } m = 0 \\ 2 & \text{if } m > 0 \end{cases}$$

From Eq. (2.19)

$$\exp[R_n(\tau) w_t w_{t'}] = \sum_{k=0}^{\infty} \frac{1}{k!} [R_n(\tau) w_t w_{t'}]^k$$

Therefore,

$$\begin{aligned}
 M_n(w_t, w_{t'}) &= \exp\left(\frac{N}{2} w_t^2\right) \exp\left(\frac{N}{2} w_{t'}^2\right) \exp[R_n(\tau) w_t w_{t'}] \\
 &= \exp\left(\frac{N}{2} w_t^2\right) \exp\left(\frac{N}{2} w_{t'}^2\right) \sum_{k=0}^{\infty} \frac{1}{k!} [R_n(\tau) w_t w_{t'}]^k
 \end{aligned}
 \tag{2.23}$$

which gives

$$\begin{aligned}
 R_y(t, t') &= \frac{1}{(2\pi j)^2} \int_C \frac{dw_t}{w_t} \int_C \frac{dw_{t'}}{w_{t'}} M_x(w_t, w_{t'}) \\
 &= \frac{1}{(2\pi j)^2} \int_C \frac{dw_t}{w_t} \int_C \frac{dw_{t'}}{w_{t'}} \left[\prod_{i=1}^p M_{s_i}(w_t, w_{t'}) \right] \\
 &\quad \cdot \exp\left(\frac{N}{2} w_t^2\right) \exp\left(\frac{N}{2} w_{t'}^2\right) \sum_{k=0}^{\infty} \frac{1}{k!} [R_n(\tau) w_t w_{t'}]^k
 \end{aligned}$$

Simplifying,

$$\begin{aligned}
 R_y(t, t') &= \frac{1}{(2\pi j)^2} \sum_{k=0}^{\infty} \frac{R_n^k(\tau)}{k!} \int_C w_t^{k-1} \exp\left(\frac{N}{2} w_t^2\right) dw_t \\
 &\quad \cdot \left[\prod_{i=1}^p M_{s_i}(w_t, w_{t'}) \right] \int_C w_{t'}^{k-1} \exp\left(\frac{N}{2} w_{t'}^2\right) dw_{t'}
 \end{aligned}
 \tag{2.24}$$

Define:

$$h_{k, m_1 \dots m_p} \triangleq \frac{1}{2\pi j} \int_C w^{k-1} \left[\prod_{i=1}^p I_{m_i}(w \sqrt{2S_i}) \right] \exp \left(\frac{N}{2} w^2 \right) dw \quad (2.25)$$

With this definition, Eq. (2.24) can be rewritten as

$$\begin{aligned} R_y(t, t') &= R_y(\tau) \\ &= \sum_{k=0}^{\infty} \sum_{m_1=0}^{\infty} \dots \sum_{m_p=0}^{\infty} \frac{R_n^k(\tau)}{k!} h_{k, m_1 \dots m_p}^2 \left(\prod_{i=1}^p \epsilon_{m_i} \cos m_i \omega_i \tau \right) \end{aligned}$$

which gives the autocorrelation function of the limiter output as

$$R_y(\tau) = \sum_{k=0}^{\infty} \sum_{m_1=0}^{\infty} \dots \sum_{m_p=0}^{\infty} \frac{R_n^k(\tau)}{k!} h_{k, m_1 \dots m_p}^2 \left(\prod_{i=1}^p \epsilon_{m_i} \cos m_i \omega_i \tau \right) \quad (2.26)$$

A product of p cosines can be expressed as a sum of cosines

$$\begin{aligned} \prod_{i=1}^p (\cos m_i \theta_i) &= \frac{1}{2^{p-1}} \left[\cos (m_1 \theta_1 + m_2 \theta_2 + \dots + m_p \theta_p) \right. \\ &\quad + \cos (m_1 \theta_1 + m_2 \theta_2 + \dots - m_p \theta_p) + \cos (m_1 \theta_1 + m_2 \theta_2 + \dots - m_{p-1} \theta_{p-1} + m_p \theta_p) \\ &\quad \left. + \cos (m_1 \theta_1 + m_2 \theta_2 + \dots - m_{p-1} \theta_{p-1} - m_p \theta_p) \dots 2^{p-1} \text{ terms} \right] \\ &\triangleq \frac{1}{2^{p-1}} [\cos (m_1 \theta_1 \pm \dots \pm m_p \theta_p)] \end{aligned} \quad (2.27)$$

where the final bracketed term represents the sum of 2^{p-1} cosines by definition.

a. Simplification of the Contour Integral $h_{k,m_1 \dots m_p}$

In evaluating the contour integral $h_{k,m_1 \dots m_p}$, the first question to resolve concerns the pole at the origin.

$$h_{k,m_1 \dots m_p} = \frac{1}{2\pi j} \int_{C_+} w^{k-1} \left[\prod_{i=1}^p I_{m_i}(w\sqrt{2S_i}) \right] \exp\left(\frac{N}{2} w^2\right) dw \\ + \frac{1}{2\pi j} \int_{C_-} w^{k-1} \left[\prod_{i=1}^p I_{m_i}(w\sqrt{2S_i}) \right] \exp\left(\frac{N}{2} w^2\right) dw \quad (2.28)$$

Note that for $w \rightarrow 0$, $I_p(w) \rightarrow w^p/2^p p!$ (see Watson, Ref. 11), the above integrand becomes

$$\frac{w^{(k+\sum m_i - 1)}}{\text{constant}} \quad \text{as } w \rightarrow 0$$

The integrand is well behaved along the whole of the $w = jv$ axis except possibly at the origin. The pole at the origin will also vanish if either k or any one of the m_i is greater than or equal to 1. If the dc term is eliminated from the covariance function, no pole will occur at the origin. All the limiter inputs have zero mean, making the integrand analytic along the entire $w = jv$ axis. The two contour integrals can now be reduced to a single integral. Hence,

$$h_{k,m_1 \dots m_p} = \frac{1}{2\pi j} 2 \int_{-j\infty}^{j\infty} w^{k-1} \left[\prod_{i=1}^p I_{m_i}(w\sqrt{2S_i}) \right] \exp\left(\frac{N}{2} w^2\right) dw$$

Converting to a real integral by using the relations $I_m(jz) = j^m J_m(z)$ and $w = jv$ yields

$$\begin{aligned}
 h_{k, m_1 \dots m_p} &= \frac{1}{\pi j} \int_{-\infty}^{\infty} j^{k-1} v^{k-1} \exp \left(-\frac{N}{2} v^2 \right) j^{\sum m_i} \prod_{i=1}^p J_{m_i} (v \sqrt{2S_i}) j \, dv \\
 &= \frac{1}{\pi} j^{(k+\sum m_i-1)} \int_{-\infty}^{\infty} v^{k-1} \exp \left(-\frac{N}{2} v^2 \right) \prod_{i=1}^p J_{m_i} (v \sqrt{2S_i}) \, dv \\
 &= \frac{1}{\pi} j^{(k+\sum m_i-1)} \left[\int_0^{\infty} v^{k-1} \exp \left(-\frac{N}{2} v^2 \right) \prod_{i=1}^p J_{m_i} (v \sqrt{2S_i}) \, dv \right. \\
 &\quad \left. - \int_{-\infty}^0 v^{k-1} \exp \left(-\frac{N}{2} v^2 \right) \prod_{i=1}^p J_{m_i} (v \sqrt{2S_i}) \, dv \right] \\
 &= \frac{1}{\pi} j^{(k+\sum m_i-1)} \left[1 - (-1)^{(k+\sum m_i)} \right] \\
 &\quad \cdot \int_0^{\infty} v^{k-1} \exp \left(-\frac{N}{2} v^2 \right) \prod_{i=1}^p J_{m_i} (v \sqrt{2S_i}) \, dv
 \end{aligned}$$

Now

$$\left[1 - (-1)^{(k+\sum m_i)} \right] = \begin{cases} 2 & \text{for } k + \sum m_i \text{ odd} \\ 0 & \text{for } k + \sum m_i \text{ even} \end{cases}$$

and hence

$$h_{k, m_1 \dots m_p} = \begin{cases} \frac{2}{\pi} \int_0^\infty v^{k-1} \exp\left(-\frac{N}{2} v^2\right) \prod_{i=1}^p J_{m_i}(v \sqrt{2S_i}) dv & \text{for } \left(k + \sum m_i\right) \text{ odd} \\ 0 & \text{for } \left(k + \sum m_i\right) \text{ even} \end{cases} \quad (2.29)$$

b. Computation of the Integral $b_{k, m_1 \dots m_p}$

It is convenient to define a quantity

$$b_{k, m_1 \dots m_p} \triangleq \left(\frac{N}{2}\right)^{k/2} h_{k, m_1 \dots m_p} \quad (2.30)$$

In the autocorrelation function, this quantity appears with an index of 2, hence the sign of the term will be dropped and the quantity redefined as

$$b_{k, m_1 \dots m_p} = \begin{cases} \frac{2}{\pi} \left(\frac{N}{2}\right)^{k/2} \int_0^\infty v^{k-1} \exp\left(-\frac{N}{2} v^2\right) \prod_{i=1}^p J_{m_i}(v \sqrt{2S_i}) dv & \text{if } \left(k + \sum m_i\right) \text{ odd} \\ 0 & \text{otherwise} \end{cases} \quad (2.31)$$

It has been found preferable to adopt the following two forms:

1. Strong Noise Case. N is greater than S_i for all $i \in (1, p)$. By an appropriate change in variable and subsequent simplification,

$$b_{k, m_1 \dots m_p} = \begin{cases} \frac{1}{\pi 2^{k-1}} \int_0^\infty x^{k-1} \exp\left(-\frac{x^2}{4}\right) \prod_{i=1}^p J_{m_i}\left[\sqrt{\frac{S_i}{N}} x\right] dx & \text{if } \left(k + \sum m_i\right) \text{ odd} \\ 0 & \text{otherwise} \end{cases} \quad (2.32)$$

2. Strong Signal Case. Let $S_1 > S_i$ for $i \in (2, p)$ and let $S_1 > N$. Then by another appropriate change in variable,

$$b_{k, m_1 \dots m_p} = \frac{1}{\pi} \frac{1}{2^{k-1}} \left(\frac{N}{S_1} \right)^{k/2} \int_0^\infty y^{k-1} \exp \left(- \frac{N}{4S_1} y^2 \right) \cdot \prod_{i=1}^p J_{m_i} \left[\left(\frac{S_i}{S_1} \right)^{1/2} y \right] dy \quad (2.33)$$

Defining $\gamma_i = \sqrt{S_i/S_1}$, then by hypothesis

$$\gamma_1 = 1 \quad \text{and} \quad \gamma_i \leq 1 \quad \text{for} \quad i \neq 1$$

hence

$$b_{k, m_1 \dots m_p} = \begin{cases} \frac{1}{\pi} \frac{1}{2^{k-1}} \left(\frac{N}{S_1} \right)^{k/2} \int_0^\infty y^{k-1} \exp \left(- \frac{N}{4S_1} y^2 \right) \prod_{i=1}^p J_{m_i}(\gamma_i y) dy & \text{if } k + \sum m_i = \text{odd} \\ 0 & \text{otherwise} \end{cases} \quad (2.34)$$

It is easy to express these forms in terms of an infinite series involving confluent, gaussian, or generalized hypergeometric functions. But it is better to deal directly with Eqs. (2.31) - (2.33) when deriving asymptotic properties and limiting values. Due care must be given to convergence problems.

For numerical results, it was found to be most convenient to carry out numerical integration on the computer directly rather than resorting to the evaluation by summation of hypergeometric series.[†]

[†]See Appendix A and Refs. 12-18 for additional formulas and tables pertaining to integrals involving Bessel functions.

c. Wideband and Narrowband Considerations

Based on the simplification and computational procedures discussed in Secs. 1a and 1b above, the following expressions that are valid for any bandwidth are obtained:

$$R_y(\tau) = \sum_{k=0}^{\infty} \sum_{m_1=0}^{\infty} \dots \sum_{m_p=0}^{\infty} \frac{R_n^k(\tau)}{k!} h_{k,m_1 \dots m_p}^2 \prod_{i=1}^p \epsilon_{m_i} \cos m_i \omega_i \tau \quad (2.35a)$$

where $R_n(\tau)$ is the autocorrelation function of the input noise.

Using the expressions and notation of Eq. (2.27), one can rewrite the above equation as

$$R_y(\tau) = \sum_{k=0}^{\infty} \sum_{m_1=0}^{\infty} \dots \sum_{m_p=0}^{\infty} \frac{R_n^k(\tau)}{k!} h_{k,m_1 \dots m_p}^2 \frac{\epsilon_{m_1} \dots \epsilon_{m_p}}{2^{p-1}} \cdot \cos [(m_1 \omega_1 \pm \dots \pm m_p \omega_p) \tau] \quad (2.35b)$$

The Fourier transform of Eq. (2.35b) gives the power spectral density of the limiter output,

$$S_y(f) = \sum_{k=0}^{\infty} \sum_{m_1=0}^{\infty} \dots \sum_{m_p=0}^{\infty} \frac{\epsilon_{m_1} \dots \epsilon_{m_p}}{k! 2^p} h_{k,m_1 \dots m_p}^2 \left[S_n(f) \overset{k}{*} S_n(f) \right] \\ * \{ \delta[f - (m_1 f_1 \pm \dots \pm m_p f_p)] + \delta[f + (m_1 f_1 \pm \dots \pm m_p f_p)] \} \quad (2.36)$$

where $\left[S_n(f) \overset{k}{*} S_n(f) \right]$ implies k convolutions of $S_n(f)$ with itself and $S_n(f)$ is the power spectral density of the input noise.

With the following points in mind, it is only a matter of straightforward computation to obtain numerical results for specific problems:

1. As k increases, the k self-convolutions of $S_n(f)$ may rapidly become gaussian.
2. $S_n(f)$ may be a reproducing density, e.g., exponential, gaussian, etc.
3. Convolutions with delta functions are particularly easy.
4. In practice, one is usually interested in a certain spectral zone and in some cases in a few terms, which simplifies matters.
5. Use should be made of the fact that $h_{k,m_1..m_p}$ exists only for the terms where $k + \sum m_i$ is odd and vanishes for the rest. Hence the terms to be computed are halved at the outset.
6. The value of $h_{k,m_1..m_p}$ may be so small as to be insignificant.

Up to this point no use has been made of the narrowband restriction. The expression for the autocorrelation of the limiter output is valid for any bandwidth.

Assuming narrowband restriction and normalizing,

$$R_n(\tau) = N\rho(\tau) \quad \text{and} \quad S_n(f) = N\sigma(f)$$

where

$$\sigma(f) = \int_{-\infty}^{\infty} \rho(\tau) e^{-i2\pi f\tau} d\tau$$

$$\rho(0) = \int_{-\infty}^{\infty} \sigma(f) df = 1$$

so that $\sigma(f)$ is a probability distribution function. The input noise has a density which is even about f_c ,

$$R_n(\tau) = N\rho(\tau) \cos \omega_c \tau$$

Substituting in Eq. (2.35b),

$$R_y(\tau) = \sum_{k=0}^{\infty} \sum_{m_1=0}^{\infty} \dots \sum_{m_p=0}^{\infty} \frac{N^k}{k!} h_{k, m_1 \dots m_p}^2 \rho^k(\tau) \cos^k(\omega_c \tau) \frac{\epsilon_{m_1} \dots \epsilon_{m_p}}{2^{p-1}} \cdot \cos [(m_1 \omega_1 \pm \dots \pm m_p \omega_p) \tau] \quad (2.37)$$

To simplify the above, use is made of the following expression [Ref. 7, page 298]:

$$\cos^k(\omega_c \tau) = \begin{cases} \frac{k!}{2^{k-1}} \sum_{i=0}^{(k-1)/2} \frac{1}{(k-i)! i!} \cos (k-2i) \omega_c \tau & \text{for } k \text{ odd} \\ \frac{k!}{2^{k-1}} \left[\sum_{i=0}^{(k-2)/2} \frac{1}{(k-i)! i!} \cos (k-2i) \omega_c \tau + \frac{1}{2 \left(\frac{k}{2}!\right)^2} \right] & \text{for } k \text{ even} \end{cases} \quad (2.38)$$

It is readily seen that the output power spectrum is centered at the following carriers:

$$(k-2i) \omega_c \pm m_1 \omega_1 \pm \dots \pm m_p \omega_p \quad (2.39)$$

Further

$$R_y(\tau) = \left\{ \begin{aligned} & \sum_{\substack{k=1 \\ k \text{ odd}}}^{\infty} \sum_{m_1=0}^{\infty} \dots \sum_{m_p=0}^{\infty} b_{k, m_1 \dots m_p}^2 \frac{\epsilon_{m_1} \dots \epsilon_{m_p}}{2^{p-1}} \rho^k(\tau) \sum_{i=0}^{(k-1)/2} \frac{1}{(k-i)! i!} \\ & \cdot \cos \{ [(k-2i)\omega_c \pm m_1\omega_1 \pm \dots \pm m_p\omega_p] \tau \} \\ & + \sum_{\substack{k=0 \\ k \text{ even}}}^{\infty} \sum_{m_1=0}^{\infty} \dots \sum_{m_p=0}^{\infty} b_{k, m_1 \dots m_p}^2 \frac{\epsilon_{m_1} \dots \epsilon_{m_p}}{2^{p-1}} \rho^k(\tau) \sum_{i=0}^{(k-2)/2} \frac{1}{(k-i)! i!} \\ & \cdot \cos \{ [(k-2i)\omega_c \pm m_1\omega_1 \pm \dots \pm m_p\omega_p] \tau \} + \frac{1}{\left(\frac{k}{2}\right)!} \\ & \cdot \cos [(m_1\omega_1 \pm \dots \pm m_p\omega_p) \tau] \quad \text{if } \left(k + \sum m_i\right) \text{ is odd} \\ & 0 \quad \text{if } \left(k + \sum m_i\right) \text{ is even} \end{aligned} \right.$$

2. Spectral Analysis and Filtering

Since the autocorrelation of the output exists for only the odd integral values of $k + \sum m_i$, the spectrum of the limiter output consists of components situated in zones around the odd harmonics of the input frequencies. As stated before, the narrowband restriction virtually eliminates interference of these output zones and hence any one zone can be selected by appropriately bandpass filtering the output.

It is now assumed that the zone around the first harmonic band around f_c is filtered. The filtered output process is denoted by $z(t)$.

The general expressions for $R_z(t)$ and $S_z(f)$ are cumbersome. The analysis is not difficult but it is lengthy and requires careful manipulations. It is better to think that the sums of cosines constituting the term $\cos^k(\omega_c \tau)$ are the carriers which are shifted up or down by the term $(m_1\omega_1 \pm \dots \pm m_p\omega_p)$. To get the first-harmonic band, the following conditions must hold:

$$k + \sum m_i = \text{odd integer} \cong 0 \quad (2.40)$$

$$(k - 2i) \pm m_1 \pm \dots \pm m_p = \pm 1$$

which implies

$$|m_1 \pm \dots \pm m_p| = \begin{cases} 0, 2, 4, \dots, k+1 & \text{for } k \text{ odd} \\ 1, 3, 5, \dots, k+1 & \text{for } k \text{ even} \end{cases}$$

Now for each of the values of m_i which satisfy the above equation, a series of terms is written. The final total of all such terms in the first-harmonic band may in some cases be expressed in a single compact form. This is possible for small values of p , such as 0, 1, or 2.

For instance, this procedure, when applied to the case $p = 2$, gives the following result which is due to Jones [Ref. 8, Eq. (13), p. 35]:

$$R_z(\tau) = \sum_{i=-\infty}^{\infty} \sum_{\ell=-\infty}^{\infty} \sum_{k=|i|, |i|+2}^{\infty} \frac{2b_k^2 |\ell| |\ell - |i+1||}{\frac{k+|i|}{2}! \frac{k-|i|}{2}!} \rho^k(\tau) \\ \cdot \cos [|i|\omega_c - |i+1|\omega_2 + \ell(\omega_2 - \omega_1)] \tau$$

where

$$b_{k,m_1 m_2} \triangleq \left(\frac{N}{2}\right)^{k/2} h_{k,m_1 m_2}$$

Similar expressions can be derived easily for $p = 0$ or 1.

It has been assumed that the input process is wide-sense stationary. If it is not weakly stationary, the only change that would result would be to replace τ by (t_1, t_2) .

3. Output Signal-to-Noise Terms

a. Output Signal Power

Fortunately it is not necessary to calculate the series for $R_z(\tau)$ to find the output signal-to-noise terms. The expression for the autocorrelation of the output consists of line spectra and continuous noise components. The periodic part of the output is primarily the result of the interaction of the input signals with themselves. The remaining terms correspond to the random variations of the output, i.e., the output noise. These terms can be split into two sets, one corresponding to the interaction of the input noise with itself and the other corresponding to input signals interacting with the input noise. Using Davenport and Root's notation [Ref. 7], $R_z(\tau)$ can be split into subsets:

$$R_z(\tau) = R_{s_1 \dots s_p}(\tau) + \sum_{i=1}^p R_{s_i n}(\tau) + \sum_{i=1}^{p-1} R_{s_i s_{i+1} n}(\tau) \dots R_{nn}(\tau) \quad (2.41)$$

where $R_{s_1 \dots s_p}(\tau)$ contains the output signal and cross terms [periodic part of $R_z(\tau)$], and $R_{nn}(\tau)$ represents the direct feedthrough noise components.

One is usually interested in only a few terms belonging to one of the $R_z(\tau)$ subsets. Hence it is not necessary to calculate the complete series of $R_z(\tau)$, which is usually extremely lengthy for values of p greater than 1.

1. The terms in $R_{s_i n}(\tau)$ are the ones that result from the interaction of the signal i with the input noise.
2. The terms in $R_{s_i s_j n}(\tau)$ are the ones that result from the interaction of both the input signals i and j with the input noise.

The very definition of these terms imposes certain obvious conditions on the k and m_i indices. These conditions, together with the need that $k + \sum m_i$ be odd, make computing the terms involved a simple process.

The subscript o will be used for output and i for input. The output signal power is given by imposing the conditions $k = 0$ and $m_j = 0$ for $j \neq i$, which give

$$(S_i)_o \bigg|_{\substack{m_i \\ \text{zone}}}^{\text{th}} = 2b_{0,00..m_i 0..0}^2 \quad (2.42)$$

Specifically, for the first zone, $m_i = 1$ gives

$$\begin{aligned} (S_i)_o \bigg|_{\substack{\text{first} \\ \text{zone}}} &= 2b_{0,00..100..0}^2 \\ &= \frac{8}{\pi^2} \left[\int_0^{(\infty)} v^{-1} \exp \left(-\frac{N}{2} v^2 \right) J_1(v \sqrt{2S_i}) \prod_{\substack{j=1 \\ j \neq i}}^p J_0(v \sqrt{2S_j}) dv \right]^2 \end{aligned} \quad (2.43)$$

b. Signal and Crossproduct Components, $R_{s_1 \dots s_p}(\tau)$

One can easily derive expressions for signal and crossproduct components in both the wideband and narrowband cases. Inspection of the expressions for $R_y(\tau)$ and $R_z(\tau)$ indicates that k must be 0. Hence, in the wideband unfiltered case,

$$\begin{aligned} y_{s_1 \dots s_p}^R(\tau) &= \sum_{m_1=0}^{(\infty)} \dots \sum_{m_p=0}^{(\infty)} b_{0,m_1 \dots m_p}^2 \left(\prod_{i=1}^p v_{m_i} \cos m_i \omega_i \tau \right) \\ &= \sum_{m_1=0}^{(\infty)} \dots \sum_{m_p=0}^{(\infty)} b_{0,m_1 \dots m_p}^2 \frac{v_{m_1} \dots v_{m_p}}{2^{p-1}} \cos [(m_1 \omega_1 + \dots + m_p \omega_p) \tau] \end{aligned}$$

when $\sum m_i$ is odd. For the narrowband case this reduces to

$$z_{s_1 \dots s_p}^R(\tau) = \sum_{m_1=0}^{\infty} \dots \sum_{m_p=0}^{\infty} b_{0, m_1 \dots m_p}^2 \frac{\epsilon_{m_1} \dots \epsilon_{m_p}}{2^{p-1}} \cos [(m_1 \omega_1 \pm \dots \pm m_p \omega_p)]$$

with the additional condition imposed on the indices that

$$m_1 \pm \dots \pm m_p = \pm 1$$

For $p = 1$ the above reduces to

$$y_s^R(\tau) = \sum_{\substack{m_1=1 \\ m_1=\text{odd}}}^{\infty} b_{0, m_1}^2 \epsilon_{m_1} \cos m_1 \omega_1 \tau = 2 \sum_{\substack{m_1=1 \\ m_1=\text{odd}}}^{\infty} b_{0, m_1}^2 \cos m_1 \omega_1 \tau$$

which is precisely the result given by Davenport and Root [Ref. 7]. For the first zone, obviously,

$$z_s^R(\tau) = 2b_{0,1}^2 \cos \omega_1 \tau$$

For any bandwidth, for $p = 2$,

$$y_{s_1 s_2}^R(\tau) = \sum_{\substack{m_1=0 \\ m_1+m_2=\text{odd}}}^{\infty} \sum_{m_2=0}^{\infty} b_{0, m_1 m_2}^2 \epsilon_{m_1} \epsilon_{m_2} \cos m_1 \omega_1 \tau \cos m_2 \omega_2 \tau$$

For the narrowband case (first-harmonic band), $z_{s_1 s_2}^R(\tau)$, the additional condition imposed on the indices is

$$m_1 \pm m_2 = \pm 1$$

Hence,

$$z^R s_1 s_2^{(\tau)} = \sum_{\substack{m_1=0 \\ m_1 \pm m_2 = \pm 1 \\ \sum m_i = \text{odd}}}^{\infty} \sum_{m_2=0}^{\infty} \frac{1}{2} \left(b_{0, m_1 m_2}^2 \epsilon_{m_1} \epsilon_{m_2} \right) \cos [(m_1 \omega_1 \pm m_2 \omega_2) \tau]$$

Further simplification in this case is possible by noting that $\cos [(m_1 \omega_1 + m_2 \omega_2) \tau]$ contributes only when

$$m_1 + m_2 = 1, \quad \text{that is,} \quad \begin{cases} m_1 = 0, & m_2 = 1 \\ \text{and} \\ m_2 = 0, & m_1 = 1 \end{cases}$$

This gives $(b_{0,10}^2 \cos \omega_1 \tau + b_{0,01}^2 \cos \omega_2 \tau)$. Also, $m_1 - m_2 = \pm 1$ implies $m_2 = m_1 \mp 1$. This contribution to the first zone due to the terms arising from $\cos [(m_1 \omega_1 - m_2 \omega_2) \tau]$ is, with appropriate restrictions,

$$\begin{aligned} & \sum_{m_1=0}^{\infty} \frac{1}{2} \left(b_{0, m_1 (m_1 \mp 1)}^2 \epsilon_{m_1} \epsilon_{m_2} \right) \cos \{ [m_1 \omega_1 - (m_1 \mp 1) \omega_2] \tau \} \\ &= \sum_{m_1=1}^{\infty} b_{0, m_1 (m_1-1)}^2 \epsilon_{m_1-1} \cos \{ [m_1 (\omega_2 - \omega_1) - \omega_2] \tau \} \\ &+ \sum_{m_1=0}^{\infty} b_{0, m_1 (m_1+1)}^2 \epsilon_{m_1} \cos \{ [-m_1 (\omega_2 - \omega_1) - \omega_2] \tau \} \\ &= \sum_{\substack{m_1=-\infty \\ m_1 \neq 1, 0}}^{\infty} 2b_{0, |m_1| |m_1-1|}^2 \cos \{ [m_1 (\omega_2 - \omega_1) - \omega_2] \tau \} + b_{0,10}^2 \cos \omega_1 \tau + b_{0,01}^2 \cos \omega_2 \tau \end{aligned}$$

Hence,

$$z R_{s_1 s_2}(\tau) = 2 \sum_{m_1=-\infty}^{\infty} b_{0, |m_1| |m_1-1|}^2 \cos \{ [m_1(\omega_2 - \omega_1) - \omega_2] \tau \} \quad (2.44)$$

This agrees with Jones' result [Ref. 8, Eq. (15), p. 36]. For higher values of p , the computations though simple and straightforward become lengthy. Fortunately, it is not necessary to derive the above series for the signal-to-noise computations. Additional conditions are imposed on the indices and these simplify matters extensively.

c. Direct Feedthrough Noise Components, $R_{nn}(\tau)$

These terms impose the condition that $m_i = 0$ for every i . Since the autocorrelation function exists only when the sum of all the indices is odd, k must be odd. Hence,

$$\begin{aligned} y R_{nn}(\tau) &= \sum_{\substack{k=1 \\ \text{odd}}}^{\infty} h_{k, \underbrace{00 \dots 0}_p}^2 \frac{N^k}{k!} \rho^k(\tau) \cos^k \omega_c \tau \\ &= \sum_{\substack{k=1 \\ \text{odd}}}^{\infty} \sum_{i=0}^{(k-1)/2} \frac{h_{k, 00 \dots 0}^2}{k!} \rho^k(\tau) \frac{k!}{2^{k-1}} \frac{1}{(k-i)! i!} \cos [(k-2i)\omega_c \tau] \\ &= \sum_{\substack{k=1 \\ \text{odd}}}^{\infty} \sum_{i=0}^{(k-1)/2} \frac{b_{k, 00 \dots 0}^2}{(k-i)! i!} \rho^k(\tau) \cos [(k-2i)\omega_c \tau] \end{aligned}$$

For $z R_{nn}(\tau)$, $(k-2i) = 1$; that is, $i = (k-1)/2$. This gives

$$z R_{nn}(\tau) = 2 \sum_{\substack{k=1 \\ \text{odd}}}^{\infty} \frac{b_{k, 00 \dots 0}^2}{\frac{k+1}{2}! \frac{k-1}{2}!} \rho^k(\tau) \cos \omega_c \tau$$

The expression reduces to Eq. (17) of Jones [Ref. 8] when $p = 2$. With no restriction placed on the bandwidth, these terms for wideband cases can be expressed as

$$R_{nn}(\tau) = \sum_{\substack{k=1 \\ \text{odd}}}^{\infty} \frac{R_n^k(\tau)}{k!} h_{k,00..0}^2$$

$$= \sum_{\substack{k=1 \\ \text{odd}}}^{\infty} \frac{N^k}{k!} \rho^k(\tau) h_{k,00..0}^2$$

Also,

$$S_{nn}(f) = \sum_{\substack{k=1 \\ \text{odd}}}^{\infty} \frac{N^k}{k!} h_{k,00..0}^2 \left[\sigma(f)^k * \sigma(f) \right]$$

C. OUTPUT FACTORS DUE TO SIGNAL AND NOISE INTERACTION

These computations are usually lengthy though straightforward. For example, for $R_{s_i n}(\tau)$, the conditions on the indices which must be met are:

- (i) $k + \sum m_i = \text{odd nonnegative integer}$
- (ii) $m_j = 0$, where $j \in (1, p)$ and $j \neq i$
- (iii) $m_i \neq 0$ and $k \neq 0$

Hence for wideband cases,

$$R_{s_i n}(\tau) = 2 \sum_{k=1}^{\infty} \sum_{\substack{m_i=1 \\ k+m_i=\text{odd}}}^{\infty} \frac{R_n^k(\tau)}{k!} h_{k,00..m_i 00..0}^2 \cos m_i \omega_i \tau$$

For the narrow bandpass case,

$$\begin{aligned}
 y_{s_i n}^R(\tau) &= 2 \sum_{k=1}^{\infty} \sum_{\substack{m_i=1 \\ k+m_i=\text{odd}}}^{\infty} \frac{N^k}{k!} h_{k,00\dots m_i 00\dots 0}^2 \rho^k(\tau) \cos^k \omega_c \tau \cos m_i \omega_i \tau \\
 &= 2 \sum_{k=1}^{\infty} \sum_{\substack{m_i=1 \\ k+m_i=\text{odd}}}^{\infty} b_{k,00\dots m_i 00\dots 0}^2 \rho^k(\tau) \sum_{j=0}^{\begin{matrix} (k-2)/2, k \text{ even} \\ (k-1)/2, k \text{ odd} \end{matrix}} \frac{1}{(k-j)! j!} \\
 &\quad \cdot \cos \{[(k-2j)\omega_c - m_i \omega_i] \tau\}
 \end{aligned}$$

Note that the $\cos \{[(k-2j)\omega_c + m_i \omega_i] \tau\}$ terms are not permissible owing to the condition on the indices.

For the first-harmonic band, further conditions are imposed,

$$\begin{aligned}
 k - 2j - m_i &= \pm 1; & k - 2j &= (m_i \pm 1) \\
 j &= \frac{k - (m_i \pm 1)}{2}; & k - j &= \frac{k + (m_i \pm 1)}{2}
 \end{aligned}$$

Also when k is even, m_i is odd, which implies $k = m_i \pm 1$. Applying this and simplifying,

$$\begin{aligned}
 z_{s_i n}^R(\tau) &= \sum_{i=-\infty}^{\infty} \sum_{k=|i|, |i|+2}^{\infty} \frac{2b_{k,00\dots |i+1| 00\dots 0}^2 \rho^k(\tau)}{\frac{k+|i|}{2}! \frac{k-|i|}{2}!} \\
 &\quad \cdot \cos \left(|i| \omega_c - (i+1) \omega_i \right) \tau
 \end{aligned}$$

This result reduces to that of Jones [Ref. 8] when $p = 2$. Similarly, expressions can be developed for the other factors of the autocorrelation function of the filtered output.

D. SMOOTH LIMITERS WITH ERROR-FUNCTION TRANSFER CHARACTERISTICS

The above results are applicable to the class of smooth limiters which have a transfer function describable by an error curve

$$g(x) = \begin{cases} g_+(x) = \operatorname{erf}(x/a) = \frac{2}{\sqrt{\pi}} \int_0^{(x/a)} e^{-\xi^2} d\xi & \text{for } x > 0 \\ 0 & \text{for } x = 0 \\ -g_+(-x) & \text{for } x < 0 \end{cases} \quad (2.45)$$

by simply replacing N in the above analysis by $(N + a^2)$, which is seen to be valid for any value of p . The value of a controls the smoothness of the limiter. For the case of one signal in noise, see Galejs [Ref. 19], Blachman [Ref. 20], and Jones [Ref. 8].

E. ASYMPTOTIC RESULTS

1. Case I, Weak Signals

Since all p signals are buried deep in noise, N is greater than S_i for all $i \in (1, p)$. From Eq. (2.3)

$$b_{0,00\dots(m_1=1)0\dots0} = \frac{2}{\pi} \int_0^\infty \frac{\exp(-x^2/4)}{x} J_1 \left[\left(\frac{S_1}{N} \right)^{1/2} x \right] \prod_{\substack{j=1 \\ j \neq 1}}^p J_0 \left[\left(\frac{S_j}{N} \right)^{1/2} x \right] dx \quad (2.46)$$

For further evaluation, use is made of the following arguments:

1. The power series representing the Bessel functions

$$J_{\mu}(z) = \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{1}{2} z\right)^{\mu+2n}}{n! (n + \mu)!}$$

is absolutely convergent for all values of x , real or complex, less than infinity.

2. Hence the power series obtained by the multiplication of a finite number of such series is also absolutely convergent for all finite values of x .

For detailed proof refer to Watson [Ref. 11, Sec. 5.41, pp. 147-148]. In particular,

$$\begin{aligned} J_1 \left[\left(\frac{S_1}{N} \right)^{1/2} x \right] \prod_{\substack{j=1 \\ j \neq i}}^p J_0 \left[\left(\frac{S_j}{N} \right)^{1/2} x \right] \\ = \frac{1}{2} \left[\left(\frac{S_i}{N} \right)^{1/2} x \right] \left\{ 1 - \left[\frac{1}{8} \left(\frac{S_i}{N} \right) + \frac{1}{4} \sum_{\substack{j=1 \\ j \neq i}}^p \left(\frac{S_j}{N} \right) \right] x^2 + \dots \right\} \\ = \frac{1}{2} \left[\left(\frac{S_i}{N} \right)^{1/2} x \right] \left(1 - c_2 x^2 + c_4 x^4 - c_6 x^6 + \dots \right) \end{aligned}$$

where c_{2n} , $n \neq 0$ are the terms involving S_i/N and its higher powers. This series is absolutely convergent. Therefore the integration and summation can be interchanged, giving

$$b_{0,00..10..0} = \frac{2}{\pi} \frac{1}{2} \left(\frac{S_i}{N} \right)^{1/2} \left[\int_0^\infty \exp \left(- \frac{x^2}{4} \right) dx \right. \\ \left. - c_2 \int_0^\infty x^2 \exp \left(- \frac{x^2}{4} \right) dx + c_4 \int_0^\infty x^4 \exp \left(- \frac{x^2}{4} \right) dx \dots \right]$$

Since each integrand exists for all $n \geq 0$,

$$b_{0,00..1..0} = \frac{1}{\pi} \left(\frac{S_i}{N} \right)^{1/2} \left[\sqrt{\pi} - c_2 \int_0^\infty x^2 \exp \left(- \frac{x^2}{4} \right) dx \dots \right]$$

Furthermore, c_2 , c_4 , etc. involve S_i/N and its higher powers, and since N can be arbitrarily chosen large enough such that the higher order terms can be neglected, then, for very large N such that $S_i/N \rightarrow 0$ for all $i \in (1,p)$,

$$b_{0,00..10..0} = \left(\frac{1}{\pi} \frac{S_i}{N} \right)^{1/2} \quad (2.47)$$

Also, under these conditions, for every $i \in (1,p)$, the output signal power in signal i in the first zone is

$$(S_i)_o \rightarrow \frac{2}{\pi} \left(\frac{S_i}{N} \right)_i \quad (2.48)$$

Hence under strong noise conditions, input signal-to-signal ratios are preserved, i.e.,

$$\left(\frac{S_j}{S_k} \right)_o = \left(\frac{S_j}{S_k} \right)_i \quad \text{for } j, k \in (1,p) \quad (2.49)$$

To compute the signal-to-noise ratio at the output, the total output noise power in this case is given by the $n \times n$ terms. The total noise power in $z(t)$, defined as N_o , is

$$N_o = R_{nn}(0) + \sum_{i=1}^p R_{s_i n}(0) + \sum R_{s_i s_{i+1} n}(0) + \dots + R_{s_1 \dots s_p}(0)$$

From Eq. (2.32) and arguments similar to the ones used above,

$$b_{k, m_1 \dots m_p} = \frac{1}{\pi 2^{k-1}} \int_0^\infty x^{k-1} \exp\left(-\frac{x^2}{4}\right) \prod_{i=1}^p \frac{[\sqrt{S_i/N} x]^{m_i}}{2^{m_i} m_i!} \left[1 - \begin{matrix} \text{higher} \\ \text{order} \\ \text{terms} \end{matrix} \right] dx$$

As $N \rightarrow \infty$ such that $S_i/N \rightarrow 0$, then $b_{k, m_1 \dots m_p}$ for any $m_i \geq 1 \rightarrow 0$ as $S_i/N \rightarrow 0$. Hence $b_{k, 00 \dots 0}$ is the only term to survive. That is, the noise at the output is contributed by $n \times n$ terms only under strong noise conditions, in which case,

$$\begin{aligned} b_{k, 00 \dots 0} &= \frac{1}{\pi 2^{k-1}} \int_0^\infty x^{k-1} \exp\left(-\frac{x^2}{4}\right) \prod_{i=0}^p J_0 \left[\left(\frac{S_i}{N} \right)^{1/2} x \right] dx \\ &\rightarrow \frac{1}{\pi 2^{k-1}} \int_0^\infty \exp\left(-\frac{x^2}{4}\right) x^{k-1} dx \\ &= \frac{1}{\pi} \int_0^\infty \exp(-t) t^{(k/2)-1} dt \end{aligned}$$

which by definition of the gamma function becomes

$$b_{k, 00 \dots 0} = \frac{1}{\pi} \Gamma\left(\frac{k}{2}\right) \quad (2.50)$$

The output noise N_o is given by

$$z_{nn}^{R}(0) = 2 \sum_{\substack{k=1 \\ \text{odd}}}^{\infty} \frac{1}{\pi^2} \frac{\Gamma^2\left(\frac{k}{2}\right)}{\frac{k+1}{2}! \frac{k-1}{2}!}$$

Substituting $k = 2j + 1$,

$$N_o = z_{nn}^{R}(0) = \frac{2}{\pi^2} \sum_{j=0}^{\infty} \frac{\Gamma^2\left(j + \frac{1}{2}\right)}{j!(j+1)!}$$

Using the generalized factorial notation given in Ref. 13,

$$\begin{aligned} z_{nn}^{R}(0) &= \frac{2}{\pi^2} \sqrt{\pi} \sqrt{\pi} \sum_{j=0}^{\infty} \frac{\left(\frac{1}{2}\right)_j \left(\frac{1}{2}\right)_j}{(2)_j j!} \\ &= \frac{2}{\pi} {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 2; 1\right) \\ &= \frac{2}{\pi} \left[E(1) \frac{4}{\pi} \right] \\ &= \frac{8}{\pi^2} \quad \text{for } N \rightarrow \infty \text{ and } S_i/N \rightarrow 0 \end{aligned} \quad (2.51)$$

In the above equation, $E(x)$ denotes the complete elliptic integral of the second kind. This result could have been predicted by observing that the limiter output is a $(-1, +1)$ square wave. Hence the total power in the fundamental zone is $(8/\pi^2)$. Since $S_i \rightarrow 0$ for every $i \in (1, p)$, $N_o \rightarrow (8/\pi^2)$. When $N \rightarrow \infty$ in such a way that $(S_i/N)_i \rightarrow 0$, the output signal-to-noise ratio is given by

$$\left(\frac{S_i}{N}\right)_0 \rightarrow \frac{1}{4} \left(\frac{S_i}{N}\right)_1 \quad \text{for any } i \in (1,p) \quad (2.52)$$

that is, a loss of 1 db approximately. Hence, when noise controls the limiter, the following general results apply:

1. All input signal-to-signal ratios are preserved when hard bandpass limiting occurs, i.e., signal suppression effect is absent.
2. The signal-to-noise ratio at the output for every signal is 1 db less than the signal-to-noise ratio at the input.

Particular cases of the above have been derived previously by Davenport for the one-signal-in-noise case [Ref. 4] and by Jones in the two-signals-in-noise case [Ref. 8].

2. Case II, One Strong Signal

a. Signal and Crossproduct Power

In this case, one signal is very strong relative to the noise and to each of the other signals. Let the strongest signal be S_1 . This is no restriction since the numbering of the input signals is arbitrary. Here

$$(S_1) \cong \sum_{i=2}^p S_i + N$$

It is also given that

$$\left(\frac{S_1}{N}\right)_1 \rightarrow \infty \quad \text{and} \quad \left(\frac{S_i}{S_1}\right)_1 \rightarrow 0 \quad \text{for} \quad \begin{cases} i \in (1,p) \\ i \neq 1 \end{cases}$$

The limiting value of the output strong signal power $(S_1)_0$ is $2b_{0,10..0}^2$, evaluated under the above given conditions. From Eq. (2.33),

$$b_{0,10..0} = \frac{2}{\pi} \int_0^\infty \frac{\exp(-Ny^2/4S_1)}{y} J_1(y) \prod_{i=2}^p J_0 \left[\left(\frac{S_i}{S_1} \right)^{1/2} y \right] dy$$

The integrand exists for all values of positive indices on y and, by the arguments developed previously,

$$b_{0,10..0} \rightarrow \frac{2}{\pi} \int_0^\infty \frac{1}{y} J_1(y) dy \quad \text{for} \quad \begin{cases} (S_i/S_1) \rightarrow 0, \quad i \neq 1, \quad \text{and} \\ (S_1/N) \rightarrow \infty \end{cases}$$

Since the integrand is 1, $b_{0,10..0} \rightarrow 2/\pi$ and therefore,

$$(S_1)_0 \rightarrow 2 \left(\frac{2}{\pi} \right)^2 = \frac{8}{\pi^2}$$

which is, as expected, the total energy in the first-harmonic zone.

An alternate way to arrive at the above is

$$\begin{aligned} b_{0,10..0} &\rightarrow \frac{2}{\pi} \int_0^\infty y^{-1} \exp \left[- \left(\frac{N}{4S_1} \right) y^2 \right] J_1(y) dy \\ &= \frac{1}{\sqrt{\pi}} \left(\frac{S_1}{N} \right)^{1/2} {}_1F_1 \left(\frac{1}{2}; 2; - \frac{S_1}{N} \right) \\ &= \frac{1}{\sqrt{\pi}} \left(\frac{S_1}{N} \right)^{1/2} \exp \left(- \frac{S_1}{2N} \right) \left[I_0 \left(\frac{S_1}{2N} \right) + I_1 \left(\frac{S_1}{2N} \right) \right] \end{aligned}$$

Noting that $I_n(z) \sim e^z / \sqrt{2\pi z}$ for large z ,

$$b_{0,10..0} \rightarrow \frac{2}{\pi}$$

which is the above result.

The limiting value of any of the weak signals at the limiter output is

$$(S_i)_0 = 2b_{0,0..(m_i=1)0..0}^2, \quad i \neq 1$$

where

$$b_{0,0 \dots (m_i=1)0 \dots 0} = \frac{2}{\pi} \int_0^\infty \frac{1}{y} \exp \left(- \frac{N}{4S_1} y^2 \right) J_1 \left[\left(\frac{S_i}{S_1} \right)^{1/2} y \right] \\ \cdot J_0(y) \prod_{\substack{j=2 \\ j \neq i}}^p J_0 \left[\left(\frac{S_j}{S_1} \right)^{1/2} y \right] dy$$

Since $(S_j/S_1)_i \rightarrow 0$ and $(N/S_1)_i \rightarrow 0$, it can easily be justified by use of the above arguments that

$$b_{0,0 \dots (m_i=1)0 \dots 0} \rightarrow \frac{2}{\pi} \int_0^\infty y^{-1} J_1 \left[\left(\frac{S_i}{S_1} \right)^{1/2} y \right] J_0(y) dy$$

This integral is the well-known Sonine and Schafheitlin integral. Hence,

$$b_{0,0 \dots (m_i=1)0 \dots 0} = \frac{2}{\pi} \frac{(S_i/S_1)^{1/2} \Gamma \left(\frac{1}{2} \right)}{2\Gamma(2)\Gamma \left(\frac{1}{2} \right)} {}_2F_1 \left[\frac{1}{2}, \frac{1}{2}; 2; \left(\frac{S_i}{S_1} \right) \right] \\ = \frac{1}{\pi} \left(\frac{S_i}{S_1} \right)^{1/2} \quad \text{as } S_i/S_1 \rightarrow 0 \quad (2.54)$$

Therefore, for $(S_1/N)_i \rightarrow \infty$ and $(S_i/S_1)_i \rightarrow 0$,

$$(S_i)_o \rightarrow \frac{2}{\pi} \left(\frac{S_i}{S_1} \right)_i \quad (2.55)$$

Hence, the output weak-signal to strong-signal power ratio is

$$\left(\frac{S_i}{S_1}\right)_0 \rightarrow \frac{1}{4} \left(\frac{S_i}{S_1}\right)_1 \quad \text{for } (S_1/N)_i \rightarrow \infty \quad \text{and} \quad (S_i/S_1)_i \rightarrow 0 \quad (2.56)$$

Therefore, if the strong signal controls the limiter, then all the other signals are modified such that the output weak-signal to strong-signal power ratio is one-fourth of the input power ratio. In the case of two signals without noise, a similar result is given [Ref. 8].

It is easy to evaluate any interfering component under the above conditions. It would be convenient to adopt the notation $S_{00..m_i 0..m_j 0..0}$ to mean the crossproduct component at a frequency $(m_i \omega_i - m_j \omega_j)$. An example would be the evaluation of the intermodulation component which occurs at a frequency of $(2\omega_1 - \omega_i)$. It is best to start with the expression for $y_{R_{s_1..s_p}}(\tau)$ in such cases. Therefore, for $(S_1/N)_i \rightarrow \infty$ and $(S_i/S_1)_i \rightarrow 0$ for $i \in (2,p)$, the ratio of the output power of the strongest intermodulation component to the output power of the weak signal is

$$\left(\frac{S_{200..i00..0}}{S_i}\right)_0 \rightarrow 1 \quad (2.57)$$

When p signals are present at the input of the limiter, such that the strongest signal S_1 controls the limiter and all the others are weak signals, the output power of the $(2\omega_1 - \omega_i)$ intermodulation component approaches that of the power of the weaker signals [Ref. 8].

b. Noise Power

To determine which of the terms in the output power contribute to the noise, the following analysis is adopted:

$$N_o = z R_{nn}(0) + z R_{s_1 n}(0) + z R_{s_2 n}(0) + \dots + z R_{s_1..s_p n}(0)$$

Using Eq. (2.33), it can easily be shown that

$$b_{k,m_1 \dots m_p} = \frac{1}{\pi} \frac{1}{2^{k-1}} \left(\frac{N}{S_1} \right)^{k/2} \int_0^\infty y^{k-1} \prod_{i=2}^p \frac{(S_i/S_1)^{m_i/2} y^{m_i}}{2^{m_i} m_i!} \cdot \left[1 - \frac{\text{higher}}{\text{order terms}} \right] J_{m_1}(y) dy$$

Since $S_i/S_1 \rightarrow 0$ if $m_i \neq 0$, $i \neq 1$,

$$b_{k,m_1 \dots m_p} \rightarrow 0 \quad \text{if any } m_i \neq 0, \quad i \in (2,p) \quad (2.58)$$

The above equation implies that the terms which contribute to the output noise power are $n \times n$ terms and $s_1 \times n$ terms and therefore

$$b_{k,m_1 00 \dots 0} \rightarrow \frac{1}{\pi 2^{k-1}} \left(\frac{N}{S_1} \right)^{k/2} \int_0^\infty y^{k-1} J_{m_1}(y) dy$$

Since $(S_1/N)_i \rightarrow \infty$ and $(S_i/S_1)_i \rightarrow 0$, the terms involving the lowest power in k , i.e., $k = 1$ are retained. The first-order component belonging to the $n \times n$ subset is

$$\begin{aligned} \frac{2b_{1,00 \dots 0}^2}{1! 0!} &\rightarrow 2 \left[\frac{1}{\pi} \left(\frac{N}{S_1} \right)^{1/2} \int_0^\infty J_0(y) dy \right]^2 \\ &= \frac{2}{\pi} \left(\frac{N}{S_1} \right)_i \end{aligned} \quad (2.59)$$

The rest of the terms in the $n \times n$ subset containing higher powers of $(N/S_1)_i$ are neglected. Also, irrespective of the value of m_1 , which must be even for $b_{1,m_1 00 \dots 0}$ to exist,

$$b_{1,m_1 00 \dots 0} \rightarrow \frac{1}{\pi} \left(\frac{N}{S_1} \right)^{1/2} \int_0^{\pi} J_{m_1}(y) dy = \frac{1}{\pi} \left(\frac{N}{S_1} \right)^{1/2}$$

Furthermore,

$$\begin{aligned} y_{s_1 n}^R(0) &= 2 \sum_{k=1}^{\infty} \sum_{m_1=1}^{\infty} \frac{N^k}{k!} h_{k,m_1 00 \dots 0}^2 \cos^k \omega_c \tau \cos m_1 \omega_1 \tau \Big|_{\tau=0} \\ z_{s_1 n}^R(0) &\Big|_{\substack{k=1, \\ m_1=2}} = 2 \left(\frac{N}{2} h_{1,200 \dots 0}^2 \right) 2 \cos \omega_c \tau \cos 2\omega_1 \tau \Big|_{\substack{\tau=0 \\ \text{filtered}}} \\ &= 2b_{1,200 \dots 0}^2 \\ &= \frac{2}{\pi} \left(\frac{N}{S_1} \right)_i \end{aligned} \quad (2.60)$$

Hence for $(S_1/N)_i \rightarrow \infty$ and $(S_i/S_1)_i \rightarrow 0$, $i \in (2,p)$, the total noise output power is

$$N_o \rightarrow 2b_{1,00 \dots 0}^2 + 2b_{1,200 \dots 0}^2 \rightarrow \frac{4}{\pi} \left(\frac{N}{S_1} \right)_i$$

The output signal-to-noise ratio for the strong signal is

$$\left(\frac{S_1}{N} \right)_o \rightarrow 2 \left(\frac{S_1}{N} \right)_i \quad (2.61)$$

The 3-db increase in the output SNR, which occurs due to hard bandpass limiting, applies only to the strongest signal, which controls

the limiter. This result for the case of one signal was given by Davenport [Ref. 4], and this analysis also checks with the one by Jones [Ref. 8].

For weak signals S_i , $i \neq 1$,

$$\left(\frac{S_i}{N}\right)_o \rightarrow \frac{1}{2} \left(\frac{S_i}{N}\right)_i \quad (2.62)$$

Each of the remaining $(p-1)$ weak signals suffers an SNR loss of 3 db when the limiter is controlled by the strong signal.

3. Case III, p Signals of Equal Strength

a. Signal Power

From Eq. (2.43) it is seen that the output signal power depends solely on $b_{k,m_1 \dots m_p}$. The condition that all the signals are of equal strength gives

$$b_{0,00 \dots (m_i=1)0 \dots 0} = \frac{2}{\pi} \int_0^\infty y^{-1} \exp\left(-\frac{N}{4S_1} y^2\right) J_1(y) J_0^{p-1}(y) dy \quad (2.63)$$

As a result of the symmetry of the integrand, it can be concluded that all signal output powers are equal;

$$(S_i)_o = \frac{8}{\pi^2} \left[\int_0^\infty y^{-1} \exp\left(-\frac{N}{4S_1} y^2\right) J_1(y) J_0^{p-1}(y) dy \right]^2$$

For the noiseless case and $p = 1$ or 2, closed-form solutions exist [Refs. 7,8]. For all other cases, solutions can be developed in the form of an infinite series involving generalized hypergeometric functions.

For large p , the following procedure culminates in a useful approximation.

$$\text{Define } I(p) = \int_0^\infty y^{-1} \exp\left(-\frac{N}{4S_1} y^2\right) J_1(y) J_0^{p-1}(y) dy$$

Comparing the series

$$J_0(y) = 1 - \frac{y^2}{2^2} + \frac{y^4}{2^2 \cdot 4^2} - \dots$$

$$J_1(y) = \frac{y}{2} \left(1 - \frac{y^2}{2 \cdot 4} + \frac{y^4}{2 \cdot 4 \cdot 4 \cdot 6} - \dots \right)$$

with

$$\exp\left[-\left(\frac{y}{2}\right)^2\right] = 1 - \frac{y^2}{2^2} + \left(\frac{y}{2}\right)^4 \frac{1}{2!} - \dots$$

$$\exp\left[-\frac{1}{2}\left(\frac{y}{2}\right)^2\right] = 1 - \frac{y^2}{2 \cdot 4} + \dots$$

and assuming p large, one obtains in the region of interest,

$$J_0(y) \approx \exp\left[-\left(\frac{y}{2}\right)^2\right] \quad (2.64a)$$

$$J_1(y) \approx \frac{y}{2} \exp\left[-\frac{1}{2}\left(\frac{y}{2}\right)^2\right] \quad (2.64b)$$

Hence,

$$I(p) \approx \int_0^\infty \frac{1}{2} \exp\left[-\left(\frac{p - \frac{1}{2} + \frac{N}{S_1}}{2}\right) \frac{y^2}{2}\right] dy$$

By a suitable change of variable,

$$I(p) \approx \left(\frac{\pi}{p - \frac{1}{2} + \frac{N}{S_1}} \right)^{1/2} \left[\frac{1}{\sqrt{2\pi}} \int_0^\infty \exp \left(-\frac{z^2}{2} \right) dz \right]$$

Since the integration has been carried out to infinity, it is preferable to replace $[p - 1/2 + (N/S_1)]$ in the denominator by $[p + (N/S_1)]$, giving

$$[I(p)]^2 \approx \frac{\pi}{4 \left(p + \frac{N}{S_1} \right)}$$

The normalized output signal power is by definition

$$\frac{(S_1)_o}{(8/\pi^2)} = - \left[1.05 + 10 \log_{10} \left(p + \frac{N}{S_1} \right) \right] \text{ in decibels} \quad (2.65)$$

This equation implies that the output signal power in decibels decreases linearly with the number of signals. Also, if the noise power is equal to the signal power, then the output signal power for p signals with noise is approximately the same as for $(p+1)$ signals without noise.

To verify the above results and also to justify the validity of the approximation, the exact output signal power was computed by numerical integration on a Philco 2000 computer and compared with the results obtained from the above approximate formula. The above formula gives results which are within 0.1 db of the exact value for p greater than or equal to 4. The results are tabulated for values of p as high as 100 (see Table 3, p. 61).

b. Crossproduct Power

The output power of, say, $S_{0..m_i 0..m_j..0}$, which has an interference frequency of $|m_i \omega_i - m_j \omega_j|$, is given by

$$\begin{aligned}
S_{0..m_i 0..m_j ..0} &= z^R s_1 s_2 \dots s_p (0) \left| \begin{array}{l} \text{appropriate} \\ \text{term} \end{array} \right. \\
&= 2 \left[\frac{2}{\pi} \int_0^\infty y^{-1} J_{m_i}(y) J_{m_j}(y) J_0^{p-2}(y) dy \right]^2 \\
&= S_{0..m_j 0..m_i ..0} \quad (2.66)
\end{aligned}$$

which by definition has an interfering frequency of $|m_j \omega_i - m_i \omega_j|$. Also note that

$$\begin{aligned}
|m_i \omega_i - m_j \omega_j|_{m_j=m_i-1} &= |m_i \omega_i - m_j \omega_j + \omega_j| \\
&= \left[\frac{\omega_i + \omega_j}{2} + \left(m_i - \frac{1}{2} \right) (\omega_i - \omega_j) \right]
\end{aligned}$$

and

$$\begin{aligned}
|m_j \omega_i - m_i \omega_j|_{m_j=m_i-1} &= |m_i \omega_i - \omega_i - m_i \omega_j| \\
&= \left[\frac{\omega_i + \omega_j}{2} - \left(m_i - \frac{1}{2} \right) (\omega_i - \omega_j) \right]
\end{aligned}$$

It can be concluded, therefore, that the output crossproduct terms situated on either side of the arithmetic mean frequency have the same power. Similarly, it can be shown that the entire output spectrum in the first zone is symmetrical about the arithmetic mean frequency.

Other results can also be derived when the inputs have equal powers, e.g.,

$$\text{Output power at } |m_{i\omega_i} - m_{j\omega_j}| = \begin{cases} \text{power at } |m_{i\omega_i} - m_{j\omega_q}| \\ \text{power at } |m_{j\omega_i} - m_{i\omega_j}| \\ \text{power at } |m_{j\omega_i} - m_{i\omega_q}| \\ \text{etc.} \end{cases}$$

Therefore, if some of the signals are of equal strength, the number of components to be computed decreases very rapidly.

For the case $p = 2$, it is easy to compute the entire series comprising the signal outputs in the first zone. The formula (see Ref. 21)

$$\int_0^\infty J_\mu(ax) J_\nu(ax) \frac{dx}{x} = \frac{2 \sin \frac{(\mu - \nu)\pi}{2}}{\pi(\mu^2 - \nu^2)}$$

$$R(\mu + \nu) > 0, \quad a > 0$$

gives

$$\begin{aligned} \left(S_{m_1 m_2} \right)_0 &= \frac{8}{\pi^2} \left\{ 2 \sum_{q=1}^{\infty} \left[\int_0^\infty \frac{J_q(y) J_{q-1}(y)}{y} dy \right]^2 \right\} \\ &= \frac{8}{\pi^2} \left[2 \sum_{q=1}^{\infty} \frac{4}{\pi^2 (2q-1)^2} \right] = \frac{64}{\pi^4} \left[\sum_{q=1}^{\infty} \frac{1}{(2q-1)^2} \right] \\ &= \frac{64}{\pi^4} \left[1 + \frac{1}{3^2} + \frac{1}{5^2} + \dots \right] \\ &= \frac{8}{\pi} \end{aligned} \tag{2.67}$$

which is, as it should be, the total energy in the first zone. This special result, which states that the power contained in the intermodulation components falls off as the reciprocals of the squares of odd integers, was given by Jones [Ref. 8]. The output power in the crossproduct $S_{\underbrace{11\dots 100\dots 0}_{\substack{q \quad p-q \\ \text{times times}}}}$ is given by

$$S_{1^q 0^{p-q}} \Big|_{\text{norm}} = \left[\frac{S_{1^q 0^{p-q}}}{\frac{8}{\pi^2}} \right] = \left[\int_0^\infty y^{-1} J_1^q(y) J_0^{p-q}(y) \exp \left(-\frac{N}{4S_1} y^2 \right) dy \right]^2 \quad (2.68)$$

The integrand is best evaluated by direct integration on a computer. To find an approximation to the above integral for large values of p , the following procedure was adopted. Equations (2.64) give

$$S_{1^q 0^{p-q}} \Big|_{\text{norm}} \approx \left\{ \int_0^\infty \exp \left(-\frac{N}{4S_1} y^2 \right) \frac{y^{q-1}}{2^q} \exp \left[-\frac{q}{2} \left(\frac{y}{2} \right)^2 \right] \cdot \exp \left[-\left(p - q \right) \left(\frac{y}{2} \right)^2 \right] dy \right\}^2$$

$$\triangleq [I_1(p)]^2$$

With an appropriate change of variable and subsequent simplification

$$I_1(p) = \frac{1}{2^q} \int_0^\infty \frac{2^{q-2} z^{(q-2)/2}}{\left(p - \frac{q}{2} + \frac{N}{S_1} \right)^{(q-2)/2}} \frac{2 dz}{p - \frac{q}{2} + \frac{N}{S_1}} e^{-z} = \frac{1}{2} \int_0^\infty \frac{z^{(q/2)-1} e^{-z}}{\left(p - \frac{q}{2} + \frac{N}{S_1} \right)^{q/2}} dz$$

Since the integration has been carried out to infinity, it is again preferable to replace $(p - q/2 + N/S_1)$ in the denominator by $(p - q/2 + 1/2 + N/S_1)$ to give

$$I_1(p) = \frac{\Gamma\left(\frac{q}{2}\right)}{2} \frac{1}{\left(p - \frac{q}{2} + \frac{1}{2} + \frac{N}{S_1}\right)^{q/2}}$$

which gives

$$S_{1q_0^{p-q}} \Big|_{\text{norm}} \approx \left[\frac{\Gamma\left(\frac{q}{2}\right)}{2} \right]^2 \frac{1}{\left(p - \frac{q}{2} + \frac{1}{2} + \frac{N}{S_1}\right)^q} \quad (2.69a)$$

$$= - \left\{ 10 \log_{10} \left[\frac{2}{\Gamma\left(\frac{q}{2}\right)} \right]^2 - 10q \log_{10} \left(p - \frac{q}{2} + \frac{1}{2} + \frac{N}{S_1} \right) \right\} \quad (2.69b)$$

in decibels

Equation (2.69) gives the following results in decibels:

For $q = 1$:

$$(S_{100..0})_o \Big|_{\text{norm}} \approx - \left[1.05 + 10 \log_{10} \left(p + \frac{N}{S_1} \right) \right] \quad (2.70a)^\dagger$$

For $q = 3$:

$$(S_{11100..0})_o \Big|_{\text{norm}} \approx - \left[7.07 + 30 \log_{10} \left(p - 1 + \frac{N}{S_1} \right) \right] \quad (2.70b)$$

For $q = 5$:

$$(S_{111110..0})_o \Big|_{\text{norm}} \approx - \left[3.55 + 50 \log_{10} \left(p - 2 + \frac{N}{S_1} \right) \right] \quad (2.70c)$$

[†]This result agrees with Eq. (2.65).

For $q = 7$:

$$\left(S_{1^7 0^{p-7}} \right)_o \Big|_{\text{norm}} \approx - \left[70 \log_{10} \left(p - 3 + \frac{N}{S_1} \right) - 4.41 \right] \quad (2.70d)$$

For $q = 9$:

$$\left(S_{1^9 0^{p-9}} \right)_o \Big|_{\text{norm}} \approx - \left[90 \log_{10} \left(p - 4 + \frac{N}{S_1} \right) - 15.29 \right] \quad (2.70e)$$

All the above expressions give results which are very close to those obtained by numerical integration on the Philco computer for large values of p . Some of the results are given in Tables 5, 6, and 7, pages 63-65.

The output power in the crossproduct $S_{2^r 1^q 0^{p-q-r}}$ is given by

$$\left. S_{2^r 1^q 0^{p-q-r}} \right|_{\text{norm}} = \left[\int_0^\infty y^{-1} dy \exp \left(- \frac{N}{4S_1} y^2 \right) J_2^r(y) J_1^q(y) J_0^{p-q-r}(y) \right]^2$$

$$\triangleq [I_2(p)]^2 \quad (2.71)$$

Proceeding on similar lines and approximating the Bessel functions in the region of interest as indicated by Eqs. (2.64) results in

$$I_2(p) \approx \int_0^\infty \frac{y^{2r+q-1}}{2^{3r+q}} dy \exp \left[- \left(\frac{p - \frac{q}{2} - \frac{2r}{3} + \frac{N}{S_1}}{4} \right) y^2 \right]$$

By a suitable change of variable and simplification,

$$I_2(p) = \frac{1}{2^{r+1} \left(p - \frac{q}{2} - \frac{2r}{3} + \frac{1}{2} + \frac{N}{S_1} \right)^{(2r+q)/2}} \Gamma \left(\frac{2r+q}{2} \right)$$

Hence ,

$$S_{2^r 1^q 0^{p-q-r}} \Big|_{\text{norm}} = \left[\frac{\Gamma\left(\frac{2r+q}{2}\right)}{2^{r+1}} \right]^2 \frac{1}{\left(p - \frac{q}{2} - \frac{2r}{3} + \frac{1}{2} + \frac{N}{S_1}\right)^{2r+q}} \quad (2.72a)$$

$$= - \left\{ 10 \log_{10} \left[\frac{2^{r+1}}{\Gamma\left(\frac{2r+q}{2}\right)} \right]^2 + 10(2r+q) \log_{10} \left(p - \frac{q}{2} - \frac{2r}{3} + \frac{1}{2} + \frac{N}{S_1} \right) \right\} \quad (2.72b)$$

in decibels

If $r = 0$, then Eq. (2.72b) reduces to Eq. (2.69).

If $r = 1$, $q = 1$, then

$$S_{2^1 1^1 0^{p-2}} \Big|_{\text{norm}} = - \left[13.09 + 30 \log_{10} \left(p - \frac{2}{3} + \frac{N}{S_1} \right) \right] \text{ in decibels} \quad (2.73)$$

If $r = 1$, $q = 3$, then

$$S_{2^1 1^3 0^{p-4}} \Big|_{\text{norm}} = - \left[9.57 + 50 \log_{10} \left(p - \frac{5}{3} + \frac{N}{S_1} \right) \right] \text{ in decibels} \quad (2.74)$$

If $r = 1$, $q = 5$, then

$$S_{2^1 1^5 0^{p-6}} \Big|_{\text{norm}} = - \left[1.61 + 70 \log_{10} \left(p - \frac{8}{3} + \frac{N}{S_1} \right) \right] \text{ in decibels} \quad (2.75)$$

If $r = 2$, $q = 1$, then

$$S_{2^2 1^1 0^{p-3}} \Big|_{\text{norm}} = - \left[15.59 + 50 \log_{10} \left(p - \frac{4}{3} + \frac{N}{S_1} \right) \right] \text{ in decibels} \quad (2.76)$$

The above expressions give values which are close to those obtained by integrating numerically on the computer. Some of the results are shown in Tables 5 and 7 on pages 63 and 65.

It may be added that such techniques can be used to get approximate expressions for the case when the signal strengths are not equal.

F. COMPUTED RESULTS

For $p \leq 2$, Jones used the technique of expanding the integrand in a series involving hypergeometric functions, integrating, and using the resulting series for computational purposes [Ref. 8]. This procedure was found to be unsuitable for values of $p > 2$. All computations were performed on the Philco 2000 computer by carrying out the appropriate integrations numerically. The numerical integration technique was also used by Shaft [Ref. 9]. Some of the results presented here follow the same format as those in Ref. 9.

The following results are presented in tabular and graphical form:[†]

1. Figures 3a and 3b show the output power in the various signal components for various $(S_1/S_3)_i$ and $(S_1/N)_i$ when $p = 3$ and when
 - a. One input signal is varied such that $S_1 \geq (S_2 = S_3)$.
 - b. Two input signals are varied such that $(S_1 = S_2) \leq S_3$.
2. Figures 4a-4c show the effect on the output power in the various signal components when $p = 4$ and when
 - a. One input signal is varied keeping the other $(p - 1) = 3$ signals equal and fixed.
 - b. Two input signals, S_1 and S_2 , are varied, keeping S_3 and S_4 constant in such a way that $(S_1 = S_2) \leq (S_3 = S_4)$.
 - c. Three input signals are varied in such a manner that $(S_1 = S_2 = S_3) \leq S_4$.
3. Figures 5a-5c show the power in the output signal components for the case of $p = 5$, under the conditions specified therein.
4. Tables 1 and 2 give the signal-suppression effect in the case of $p = 3$ and $p = 4$.
5. Figures 6a-6b and 7a-7c show the signal-suppression effect graphically.

[†]Figures 3-9 and Tables 1-7 appear at the end of this chapter beginning on p. 54.

6. Table 3 gives the output signal power in the case of p equal signals (no noise) for values of $1 \leq p \leq 100$. Both computed and approximate values using Eq. (2.65) are given.
7. Table 4 gives the output signal power in the case of p equal signals in noise for $3 \leq p \leq 100$.
8. Table 5 gives the strength of some of the crossproducts for various input signal-to-noise ratios in the case of four equal signals.
9. Table 6 gives the crossproducts for p equal signals in noise for large p .
10. Table 7 gives other crossproducts for the case of p equal signals (no noise).
11. Figures 8 and 9 display the output power in the crossproducts for large p in the equal-signal case and clearly show that the values obtained by the approximate formulas are indiscernible from the exact values obtained through computer evaluation of the exact expressions.

G. CONCLUSIONS TO CHAPTER II

It has been shown that many of the limiting properties for the case of hard limiting of p inputs in random noise are the same as those for the case of $p = 2$ under similar conditions. The following conclusions are in order:

1. The limits on signal-to-noise ratio are:

For one signal in noise,

$$\frac{\pi}{4} \leq \frac{(S/N)_o}{(S/N)_i} \leq 2$$

For p signals in noise,

$$0 \leq \frac{(S/N)_o}{(S/N)_i} \leq 2$$

2. For the strong-noise case, the following results which are known for $p = 1, 2$ [Ref. 8], have been shown to be applicable for any p :
 - a. The output signal-to-signal power ratios are the same as the input signal-to-signal power ratios for any number of inputs, that is, there is an absence of signal suppression.
 - b. For any p the signal-to-noise ratio at the output is 1 db less than the signal-to-noise ratio at the input.
3. For one strong signal S_1 and $(p-1)$ weak signals S_i :
 - a. If the input (S_1/S_i) power ratio is very large, the behavior approaches that of the one-signal-in-noise case as the ratio of the input strong signal to the weak signal increases, and

$$S_1 \gg \sum_{i=2}^p S_i + N$$

Then, for large input signal-to-noise ratios:

- (1) The range of the normalized SNR's approaches the one-signal-in-noise case.
- (2) The suppression effect manifests itself, with the amount approaching the limiting value,

$$(S_1/S_i)_o \rightarrow 4(S_1/S_i)_i, \quad \text{that is, 6 db.}$$
- (3) The strong signal receives an improvement in relative SNR, which approaches a value of 2, that is, 3 db.
- (4) Each of the $(p-1)$ signals suffers a loss in SNR by a factor of 2, that is, 3 db.
- b. If the ratio $(S_1/S_i)_i$ is larger than approximately 20/1 and for $(S_1/N)_i$ greater than 1, the effects of limiting are the same as the equivalent two-signals-in-noise case.
- c. If the ratio $(S_1/S_i)_i$ is greater than or equal to 20/1, and that for $(S_1/N)_i$ is less than 1, then for $p > 2$, the suppression effect becomes less pronounced and ultimately the weak signals are boosted instead of being suppressed.

4. For two strong signals, say S_1 and S_2 , and $(p-2)$ weak signals S_i , the behavior follows the patterns set by the two-signals-in-noise case. There is one exception, however, which is the boosting of the weak signals for $(S_2/S_1)_i > 20/1$ and $(S_2/N)_i < 1$. These are the very same conditions as for item 3d. The strong signals cause a beat pattern and every time the envelope of the two strong signals falls below the noise or weak signal, the latter gets control of the limiter. Consequently, all the weak signals get transmitted without any attenuation. The effect is similar to the one described by Jones [Ref. 8], which is an attempt to give some physical explanation as to the decrease in the relative signal-to-noise ratios for high input signal-to-noise ratios.[†]
5. In all cases the crossproducts follow the curves in Fig. 4 of Jones [Ref. 8] very closely. The signal power in the crossproducts remains nearly constant for input SNR's greater than 6 and decreases very rapidly for smaller input SNR's.
6. From the numerical results and graphs presented, it is concluded that the power in the output components remains relatively independent of the input SNR's for input strong-signal-to-noise ratios greater than 10 db.
7. For large p , and approximately equal signals, Eqs. (2.70) through (2.76) are valid.
8. For p large, the following results were obtained:
 - a. The power in any signal or crossproduct term decreases in a linear fashion when plotted on a log-log basis [see Eqs. (2.70) - (2.76)].
 - b. The sum of the powers in all the p signals at the output is independent of p . From Eq. (2.70),

$$p(S_{100\dots 0})_o \Big|_{\text{norm}} = -1.05 + \log_{10} \left(p + \frac{N}{S_1} \right) + \log_{10}(p)$$

$$\approx -1.05 \text{ db} \quad (2.77)$$

[†]Other cases may be reduced in a similar manner to predict the effects of limiting.

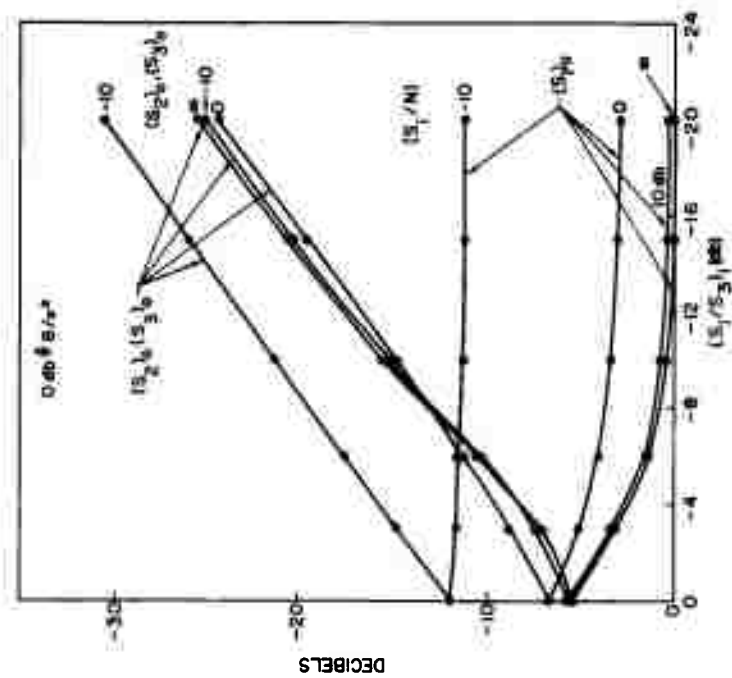
that is, the power contained in the p signals is approximately 1.1 db less than the total power in the output first-harmonic zone.

- c. The p -signals-in-noise case has approximately the same behavior due to hard limiting as the case of $[p + (N/S_1)]$ signals without noise.
- d. The power in decibels in any term at the output decreases as

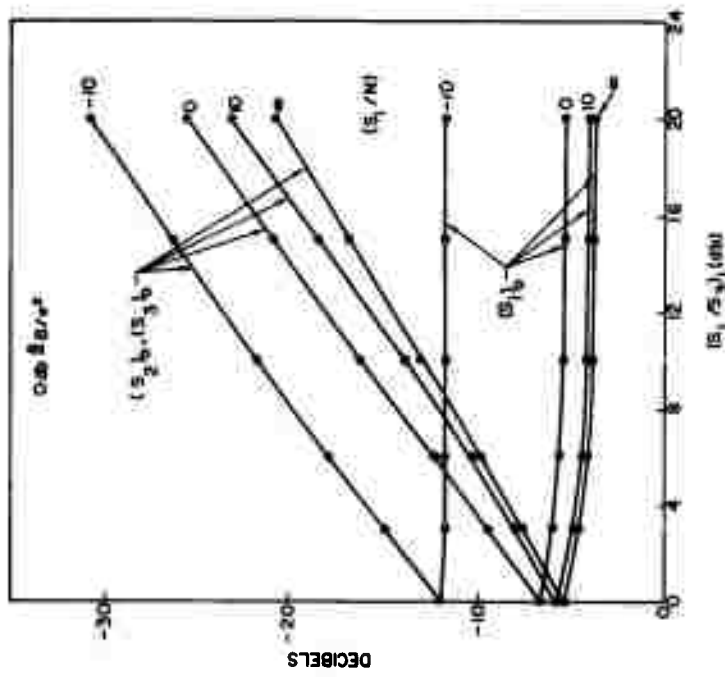
$$-10 \left[\sum_{i=1}^p m_i \right] \log_{10} p \quad (2.78a)$$

that is,

$$\text{Output power} \sim p^{\left(\sum_{i=1}^p m_i \right)} \quad (2.78b)$$

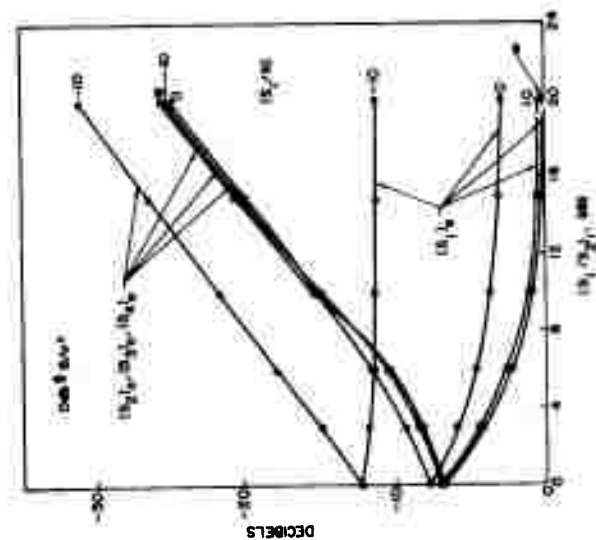


a. $S_1 \geq (S_2 = S_3)$

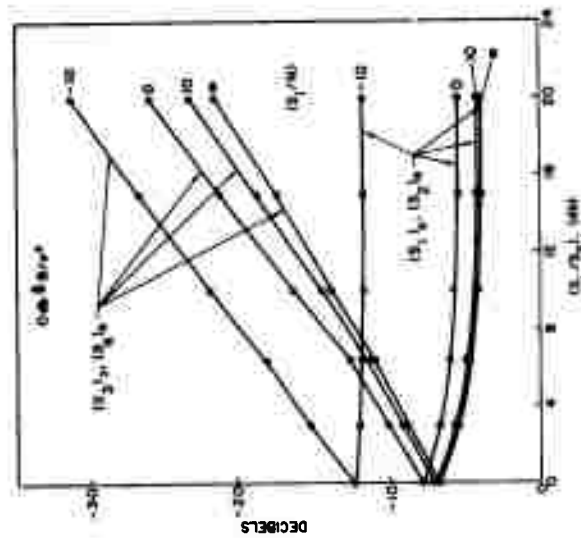


b. $(S_1 = S_2) \geq S_3$

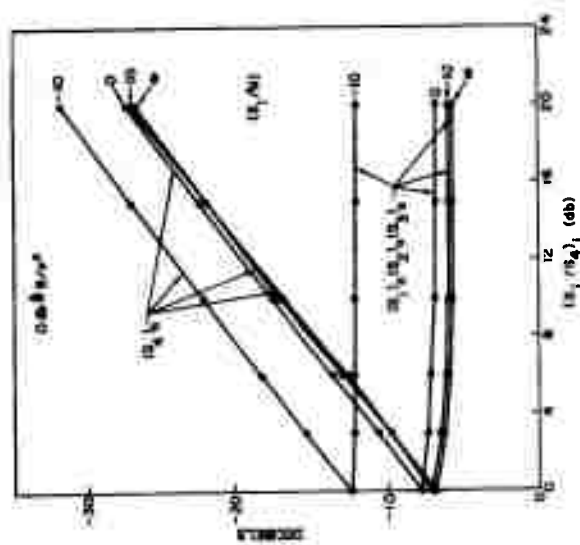
FIG. 3. OUTPUT SIGNAL POWER FOR THE THREE-SIGNAL CASE.



a. $S_1 \geq (S_2 = S_3 = S_4)$

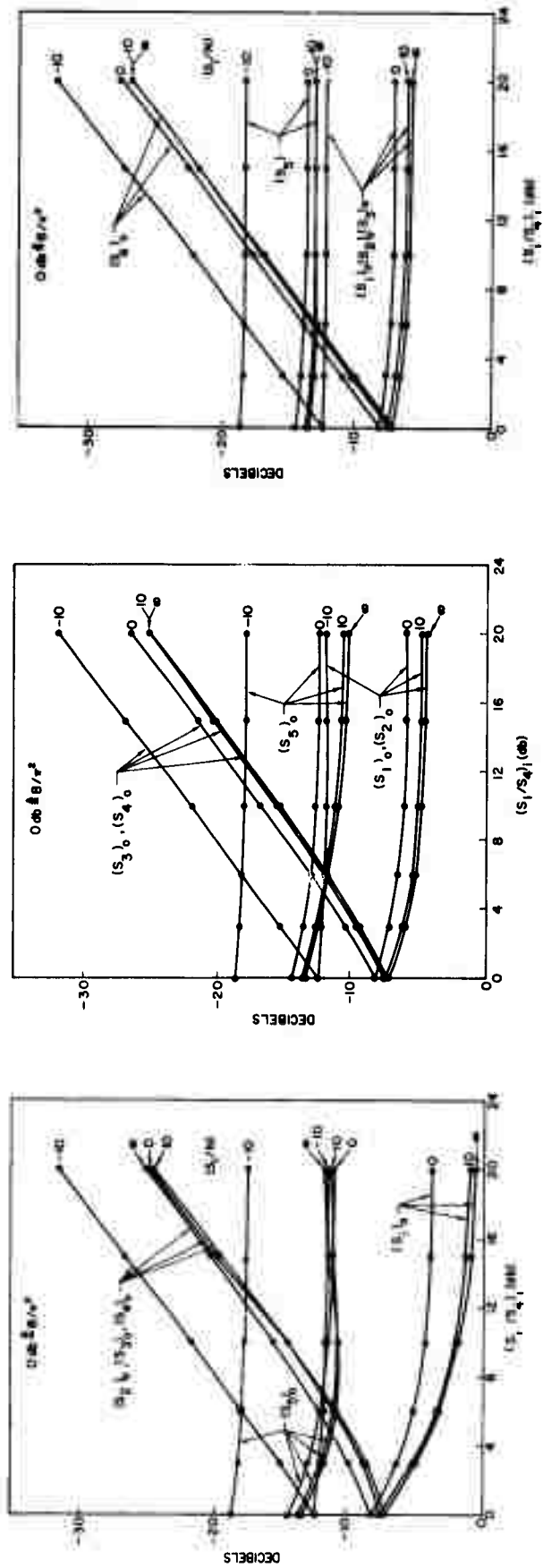


b. $(S_1 = S_2) \geq (S_3 = S_4)$



c. $(S_1 = S_2 = S_3) \geq S_4$

FIG. 4. OUTPUT SIGNAL POWER FOR THE FOUR-SIGNAL CASE.



a. $S_1 \geq (S_2 = S_3 = S_4)$, $S_1 = 4S_5$ b. $(S_1 = S_2) \geq (S_3 = S_4)$, $S_1 = 4S_5$ c. $(S_1 = S_2 = S_3) \geq S_4$, $S_1 = 4S_5$

FIG. 5. OUTPUT SIGNAL POWER FOR THE FIVE-SIGNAL CASE.

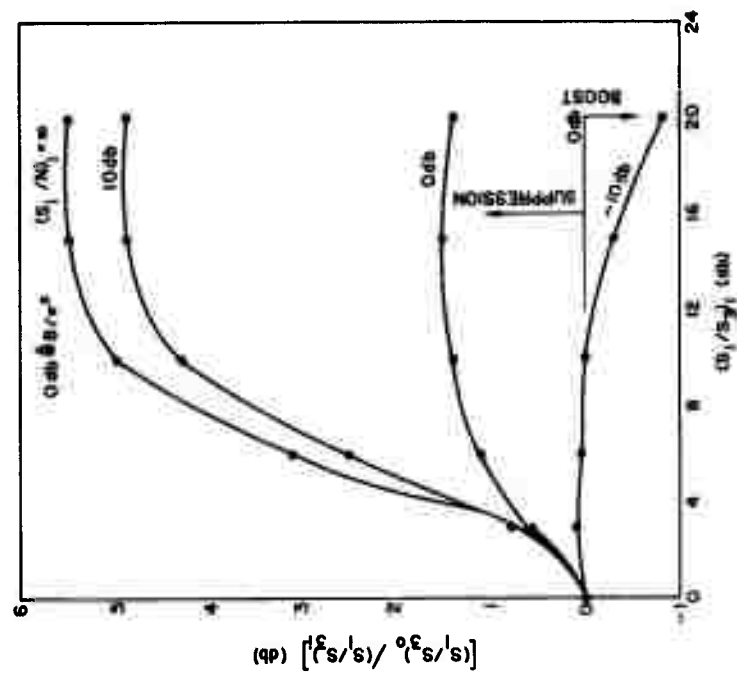
TABLE 1. SUPPRESSION EFFECT FOR THE THREE-SIGNAL CASE ($p = 3$)
(All values in decibels)

$\left(\frac{S_1}{S_3}\right)_i$	$(S_1/N)_i = \infty$		$(S_1/N)_i = 10$		$(S_1/N)_i = 0$		$(S_1/N)_i = -10$	
	$\left(\frac{S_1}{S_3}\right)_o$	$\left[\frac{(S_1/S_3)_o}{(S_1/S_3)_i}\right]$	$\left(\frac{S_1}{S_3}\right)_o$	$\left[\frac{(S_1/S_3)_o}{(S_1/S_3)_i}\right]$	$\left(\frac{S_1}{S_3}\right)_o$	$\left[\frac{(S_1/S_3)_o}{(S_1/S_3)_i}\right]$	$\left(\frac{S_1}{S_3}\right)_o$	$\left[\frac{(S_1/S_3)_o}{(S_1/S_3)_i}\right]$
Condition $S_1 = (S_2 = S_3)$								
0	0	0	0	0	0	0	0	0
3	3.6	0.6	3.8	0.8	3.6	0.6	3.1	0.1
6	9.1	3.1	8.5	2.5	7.1	1.1	6.0	0
10	15.0	5.0	14.3	4.3	11.4	1.4	10.0	0
15	20.5	5.5	19.9	4.9	16.5	1.5	14.7	-0.3
20	25.5	5.5	24.9	4.9	21.4	1.4	19.2	-0.8
Condition $(S_1 = S_2) > S_3$								
0	0	0	0	0	0	0	0	0
3	2.9	-0.1	3.1	0.1	3.4	0.4	3.1	0.1
6	5.5	-0.5	5.9	-0.1	6.5	0.5	6.1	0.1
10	8.9	-1.1	9.5	-0.5	10.5	0.5	9.9	-0.1
15	12.7	-2.3	14.2	-0.8	15.4	0.4	14.6	-0.4
20	16.8	-3.2	19.0	-1.0	20.1	0.1	19.1	-0.8

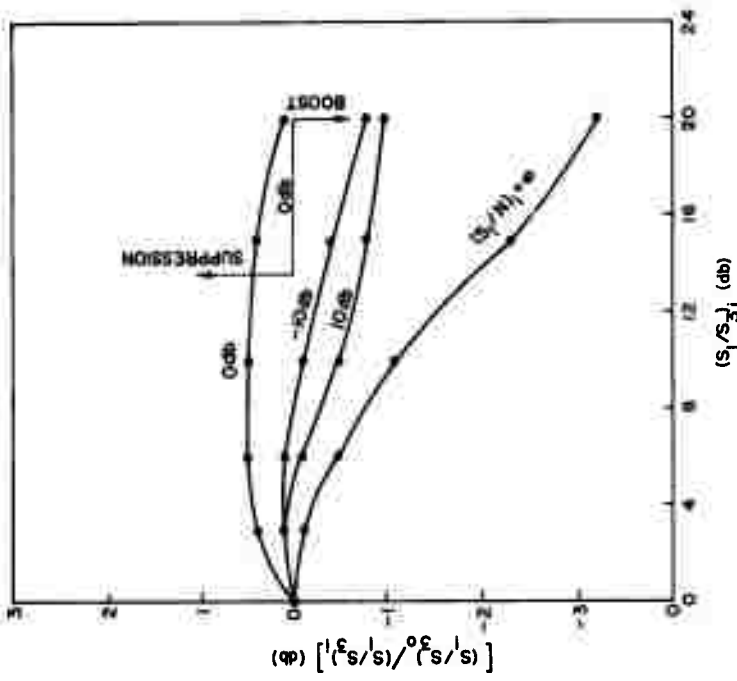
TABLE 2. SUPPRESSION EFFECT FOR THE FOUR-SIGNAL CASE ($p = 4$)

(All values in decibels)

$\left(\frac{S_1}{S_4}\right)_i$	$(S_1/N)_i = \infty$		$(S_1/N)_i = 10$		$(S_1/N)_i = 0$		$(S_1/N)_i = -10$	
	$\left(\frac{S_1}{S_4}\right)_o$	$\left[\frac{(S_1/S_4)_o}{(S_1/S_4)_i}\right]$	$\left(\frac{S_1}{S_4}\right)_o$	$\left[\frac{(S_1/S_4)_o}{(S_1/S_4)_i}\right]$	$\left(\frac{S_1}{S_4}\right)_o$	$\left[\frac{(S_1/S_4)_o}{(S_1/S_4)_i}\right]$	$\left(\frac{S_1}{S_4}\right)_o$	$\left[\frac{(S_1/S_4)_o}{(S_1/S_4)_i}\right]$
Condition $S_1 < (S_2 = S_3 = S_4)$								
0	0	0	0	0	0	0	0	0
3	3.8	0.8	3.7	0.7	3.4	0.4	3.1	0.1
6	7.9	1.9	7.8	1.8	6.9	0.9	6.2	0.2
10	14.4	4.4	13.6	3.6	11.4	1.4	10.1	0.1
15	20.3	5.3	19.7	4.7	16.6	1.6	15.0	0.0
20	25.3	5.3	24.9	4.9	21.7	1.7	19.8	-0.2
Condition $(S_1 = S_2) < (S_3 = S_4)$								
0	0	0	0	0	0	0	0	0
3	3.4	0.4	3.5	0.5	3.4	0.4	3.1	0.1
6	6.2	0.2	6.4	0.4	6.5	0.5	6.2	0.2
10	9.5	-0.5	10.0	0.0	10.6	0.6	10.1	0.1
15	13.4	-1.6	14.4	-0.6	15.6	0.6	14.9	-0.1
20	17.5	-2.5	19.1	-0.9	20.4	0.4	19.7	-0.3
Condition $(S_1 = S_2 = S_3) < S_4$								
0	0	0	0	0	0	0	0	0
3	3.7	0.7	3.5	0.5	3.3	0.3	3.1	0.1
6	7.1	1.1	6.9	0.9	6.4	0.4	6.1	0.1
10	11.3	1.3	11.1	1.1	10.5	0.5	10.0	0.0
15	16.3	1.3	16.2	1.2	15.5	0.5	15.0	0.0
20	21.1	1.1	21.1	1.1	20.3	0.3	19.8	-0.2



a. $S_1 \geq (S_2 = S_3)$



b. $(S_1 = S_2) > S_3$

FIG. 6. SUPPRESSION EFFECT FOR $p = 3$.

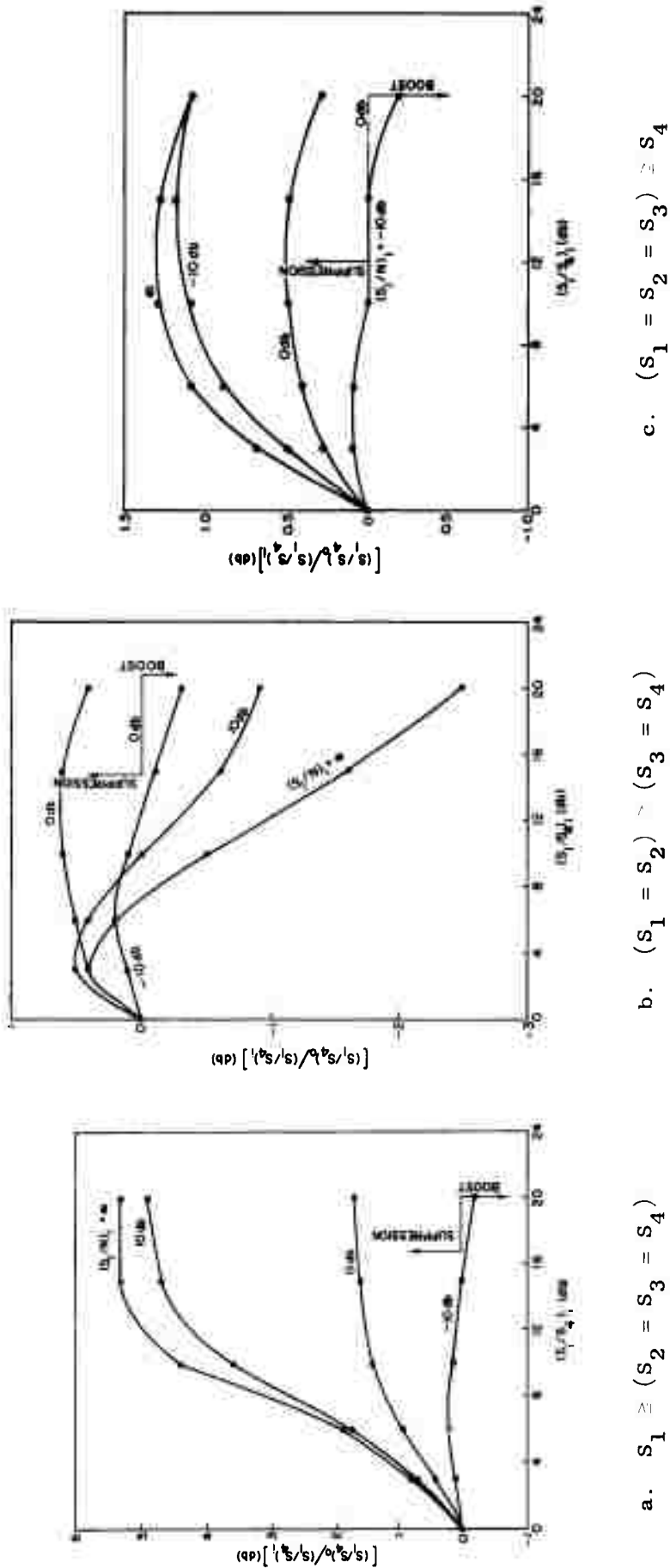


FIG. 7. SUPPRESSION EFFECT FOR $p = 4$.

TABLE 3. OUTPUT SIGNAL POWER FOR p EQUAL SIGNALS AND NO NOISE

Total energy in first zone = $8/\pi^2 = 0$ db
(All computed values* in minus decibels)

Number of Signals (p)	Exact Value	Approximate Value	Number of Signals (p)	Exact Value	Approximate Value
1	0	1.05	14	12.5	12.5
2	3.9	4.1	15	12.8	12.8
3	5.6	5.8	16	13.1	13.1
4	6.9	7.1	17	13.3	13.4
5	7.9	8.0	18	13.6	13.6
6	8.7	8.8	19	13.8	13.8
7	9.4	9.5	20	14.1	14.1
8	10.1	10.1	30	15.9	15.8
9	10.6	10.6	40	17.1	17.1
10	11.0	11.1	50	18.1	18.0
11	11.5	11.5	75	19.9	19.8
12	11.8	11.8	100	21.2	21.1
13	12.2	12.2			

*Values: Exact $= (S_i)_0 \left[\int_0^\infty y^{-1} J_1(y) J_0^{p-1}(y) dy \right]^2$.

Approximate = $-[1.05 + 10 \log_{10} (p + N/S_i)]$,

N = 0, noiseless case.

Note: For all values of p greater than 4, the approximate formula gives the answers to within 0.1 db. For the most part the error is due to rounding off the values to the first place.

TABLE 4. OUTPUT SIGNAL POWER FOR p EQUAL SIGNALS IN NOISE
(All computed values* in minus decibels)

Number of Signals (p)	$(S/N)_i = 10 \text{ db}$		$(S/N)_i = 0 \text{ db}$		$(S/N)_i = -10 \text{ db}$		$(S/N)_i = -20 \text{ db}$	
	Exact	Approximate	Exact	Approximate	Exact	Approximate	Exact	Approximate
3	5.7	5.1	6.8	7.1	12.0	12.2	--	--
4	7.0	7.2	7.8	8.0	12.4	12.5	--	--
8	10.1	10.1	10.6	10.6	13.6	13.6	21.7	21.4
10	11.1	11.1	11.5	11.5	14.1	14.1	21.8	21.5
20	14.2	14.1	14.4	14.3	16.0	15.8	22.2	21.9
40	17.3	17.1	17.4	17.2	18.2	18.0	22.9	22.5
70	19.8	19.5	19.8	19.6	20.4	20.1	23.8	23.4
100	21.4	21.1	21.4	21.1	21.8	21.5	24.5	24.1

$$^* \text{Values: } \text{Exact} = (S_i)_o \Big|_{\text{norm}} = \left[\int_0^\infty y^{-1} J_1(y) J_0^{p-1}(y) \exp \left(-\frac{N}{4S_1} y^2 \right) dy \right]^2$$

$$\text{Approximate} = -[1.05 + 10 \log_{10} (p + N/S_1)]$$

TABLE 5. CROSSPRODUCTS VS (S/N) FOR THE FOUR-SIGNAL CASE

$$S_1 = S_2 = S_3 = S_4$$

(All computed values in minus decibels)

<p>Normalization = $8/\pi^2 = 0$ db</p> <p>Notation: $S_{1000} = S_{1^1 0^3}$ = signal at frequency $1f_1$</p> <p>$S_{2100} = S_{2^1 1^1 0^2}$ = power at frequency $(2f_1 \pm f_2)$</p> <p>$S_{2111} = S_{2^1 1^3}$ = power at frequency $(2f_1 \pm f_2 \pm f_3 \pm f_4)$</p>						
Crossproduct	(S/N) _i					
	∞	20 db	15 db	10 db	5 db	0 db
$S_{1^1 0^3}$	6.9	6.9	6.9	7.0	7.2	7.9
$S_{2^1 1^1 0^2}$	28.5	28.5	28.6	28.9	29.7	31.9
$S_{1^3 0^1}$	21.5	21.5	21.6	21.8	22.5	24.7
$S_{2^3 1^1}$	31.6	31.7	32.1	33.0	35.9	43.7

TABLE 6. CROSSPRODUCTS FOR LARGE p IN NOISE (EQUAL-SIGNAL CASE)
(All computed values in minus decibels)

Normalization = $8/\pi^2 = 0 \text{ db}$									
$\text{Values: Exact} = S_{11100..0}^{3 \text{ p-3}} \Big _{\text{norm}} = S_{10^{3 \text{ p-3}}} \Big _{\text{norm}} = \left[\int_0^\infty J_1^3(y) J_0^{p-3}(y) \exp\left(-\frac{N}{4S_1} y^2\right) \frac{dy}{y} \right]^2$									
$\text{Approximate } \triangleq - \left[7.07 + 30 \log_{10} \left(p - 1 + \frac{N}{S_1} \right) \right] \text{ db}$									
Number of Signals (p)	(S/N) _i = ∞		(S/N) _i = 10 db		(S/N) _i = 0 db		(S/N) _i = -10 db		
	Exact	Approximate	Exact	Approximate	Exact	Approximate	Exact	Approximate	
10	35.8	35.7	35.9	35.8	37.1	37.1	45.5	45.4	
20	45.5	45.4	45.5	45.5	46.1	46.1	50.9	50.9	
40	54.8	54.8	54.9	54.8	55.1	55.1	57.8	57.8	
100	66.9	66.9	66.9	67.0	67.1	67.1	68.2	68.2	

The approximate formula, Eq. (2.70b), gives excellent results.

TABLE 7. CROSSPRODUCTS FOR LARGE p , NO NOISE (EQUAL-SIGNALS CASE)

Normalization = $8/\pi^2 = 0$ db
(All computed values in minus decibels)

Exact Value: Eq. (2.68a); Approximate Value: Eq. (2.68b)						
Number of Signals (p)	$S_{10}p-5$		$S_{10}p-7$		$S_{10}p-9$	
	Exact	Approximate Eq. (2.70c)	Exact	Approximate Eq. (2.70d)	Exact	Approximate Eq. (2.70e)
10	49.3	48.7	56.1	54.8	56.4	54.8
20	66.7	66.3	82.6	81.7	94.7	93.1
30	76.1	75.9	96.4	95.8	113.1	112.1
40	82.7	82.5	105.8	105.4	125.6	124.8
50	87.7	87.6	113.0	112.6	135.0	134.4
100	103.2	103.1	134.8	134.7	163.4	163.1

The approximation gives good results in every case.

Exact Value: Eq. (2.71); Approximate Value: Eq. (2.72a)						
Number of Signals (p)	$S_{21}p-2$		$S_{21}p-4$		$S_{21}p-6$	
	Exact	Approximate Eq. (2.73)	Exact	Approximate Eq. (2.74)	Exact	Approximate Eq. (2.75)
10	42.3	42.2	56.3	55.6	63.8	63.3
20	51.7	51.7	73.1	72.7	89.2	79.5
30	57.1	57.1	82.4	82.2	102.8	88.7
40	61.0	60.9	88.9	88.8	112.8	95.1
50	63.9	63.9	93.9	93.8	119.2	100.1
100	73.0	73.0	109.3	109.2	141.0	115.4
						Approximate Eq. (2.76)
						62.5
						79.2
						88.5
						95.0
						100.0
						115.3

The approximate formulas are justified for large p .

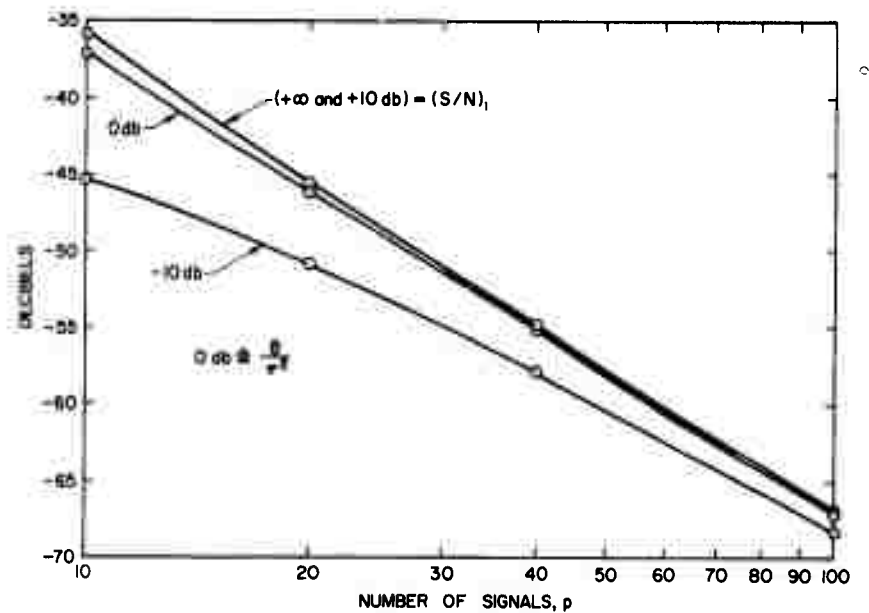
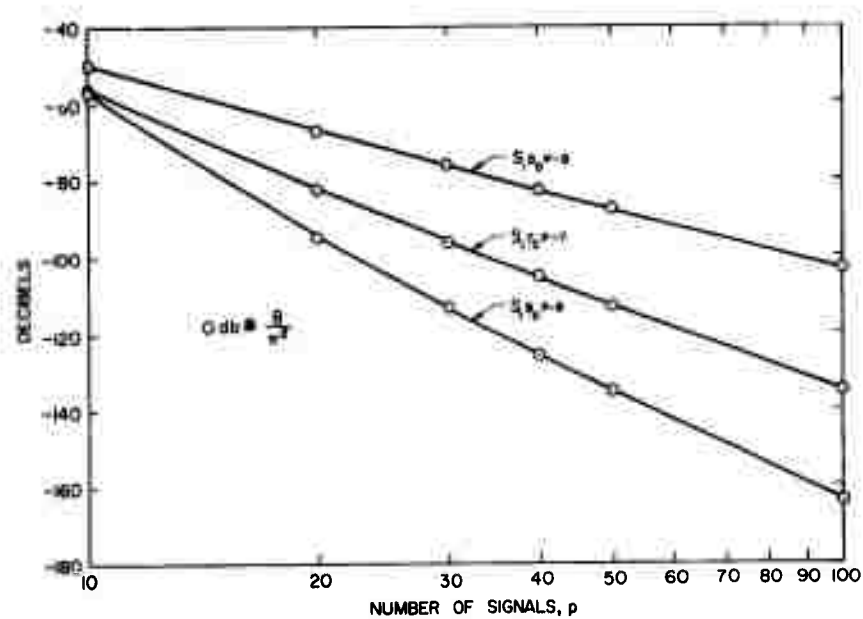


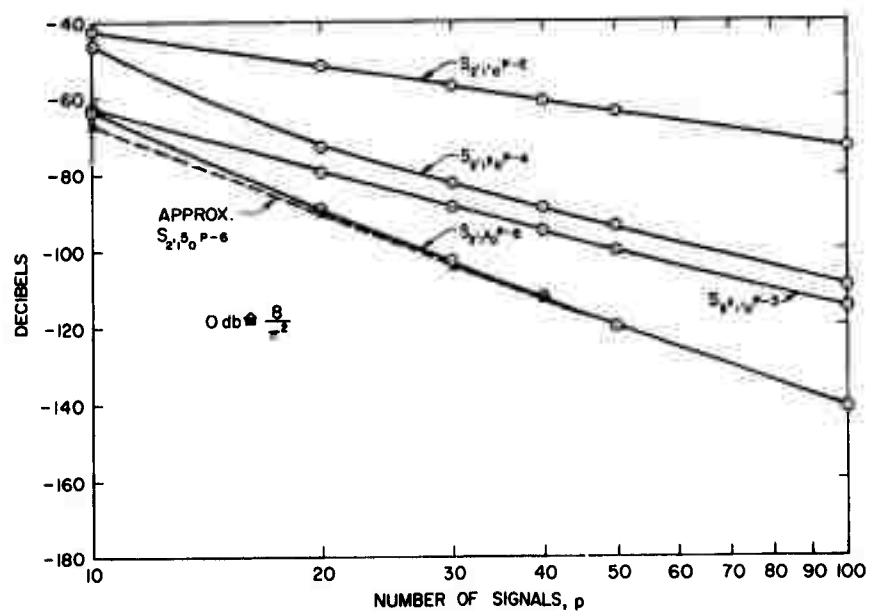
FIG. 8. CROSSPRODUCT S_{10}^{3p-3} VS p FOR

EQUAL-SIGNALS-IN-NOISE CASE.

In Figs. 8 and 9 where the approximate curve does not coincide with the exact curve, it is indicated by a dashed line.



a. $S_{10}^{p-5}, S_{10}^{p-7}, S_{10}^{p-9}, \left(S_{10}^{p-q} \right)$



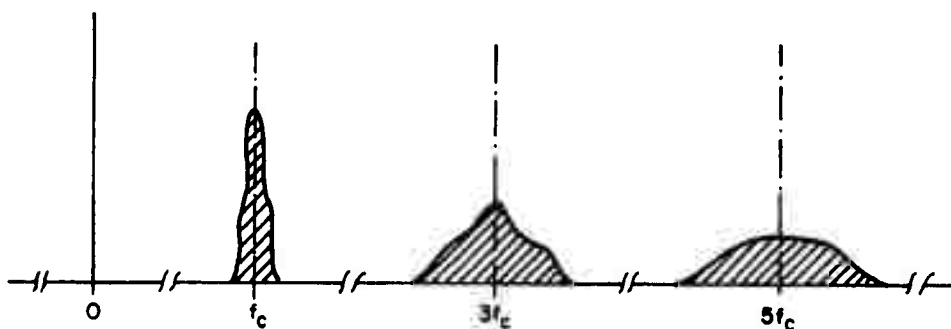
b. S_{21}^{p-q-r}

FIG. 9. CROSSPRODUCTS VS p FOR EQUAL-SIGNALS, NO-NOISE CASE.

III. OUTPUT SPECTRA FOR WIDEBAND FM PULSE TRAINS GENERATED FROM HARD-LIMITED FM SIGNALS

A. INTRODUCTION

In most narrowband FM systems the nominal carrier frequency is much larger than the maximum information bandwidths, e.g., an FM broadcast channel bandwidth is approximately 36 kc, while the carrier is a few megacycles. In the process of demodulation, the received signal is subjected to hard limiting. The amount of energy in the sideband of the third harmonic which lies in the first-harmonic band is typically less than 10^{-6} of the energy in the fundamental band. Hence the output of the limiter, to all intents and purposes, consists of discrete spectral zones as shown in the following sketch; which is a typical spectrum of a limited narrowband FM signal. When such a signal is passed through a narrow bandpass filter, the fundamental band can be filtered out. Since it is essential that such a bandpass filter have an excellent time-delay characteristic, the filtered fundamental zone does not suffer any appreciable phase distortion, nor is the spectral distribution of the first zone disturbed. As a result, the filtered output has no envelope structure. Therefore this signal can be demodulated by means of a relatively narrowband discriminator, which may be a tuned circuit or a balanced detector of the conventional type [Ref. 22]. Such systems have been extensively analyzed by various investigators [e.g., Middleton, Ref. 2, and Loughlin, Ref. 23].



As an illustration, consider a case of monotone frequency modulation where the normalized carrier frequency is 1 and the modulating frequency is approximately 0.5. Assume further that the deviation is ± 50 percent. The limited spectral output is as shown in Fig. 10a. It is clear that in order to filter out the fundamental zone immediately prior to detection, as before, it would be preferable in this case to use a lowpass filter of bandwidth $2(\Delta f + f_m) = 2$ [Refs. 24 and 25]. The filtered output will then be as shown in Fig. 10b. Because of the high selectivity of the lowpass filter, excessive phase distortions in the sidebands of the filtered output will occur. In addition, since the filtered output contains quite a few significant sidebands of the third-harmonic zone, an undesirable envelope structure will be present. The detected output may be computed using the techniques given in Refs. 24 and 26. In any case, since the amplitudes of the in-band third-harmonic lower sidebands are appreciable, the detected output will certainly have unacceptable spurious components. Hence, filtering the limited output is not desirable in low-carrier wide-deviation systems.[†]

Since the spurious outputs usually limit the system performance, it has been necessary to define an "interference ratio" (see Sec. E). A complete analysis of such a demodulation scheme is covered in this chapter. The autocorrelation function and the power spectral density of the output process, as well as closed-form expressions for the interference ratio, are derived for the general case. The closed-form expressions are used to compute the spurious outputs which are within 2 db of the experimentally measured values. The specifications of some of the demodulators designed on the basis of this analysis are also presented.

B. INPUT PROCESS

1. Assumption

It is assumed that the limiter has a sufficiently wide bandwidth, which would imply that the output of the limiter can be considered to be

[†]For these systems, satisfactory performance may be obtained through detection by means of integrating a train of arbitrary but fixed, unidirectional pulses which are generated every time the limited wave passes through zero [Cady, Ref. 27].

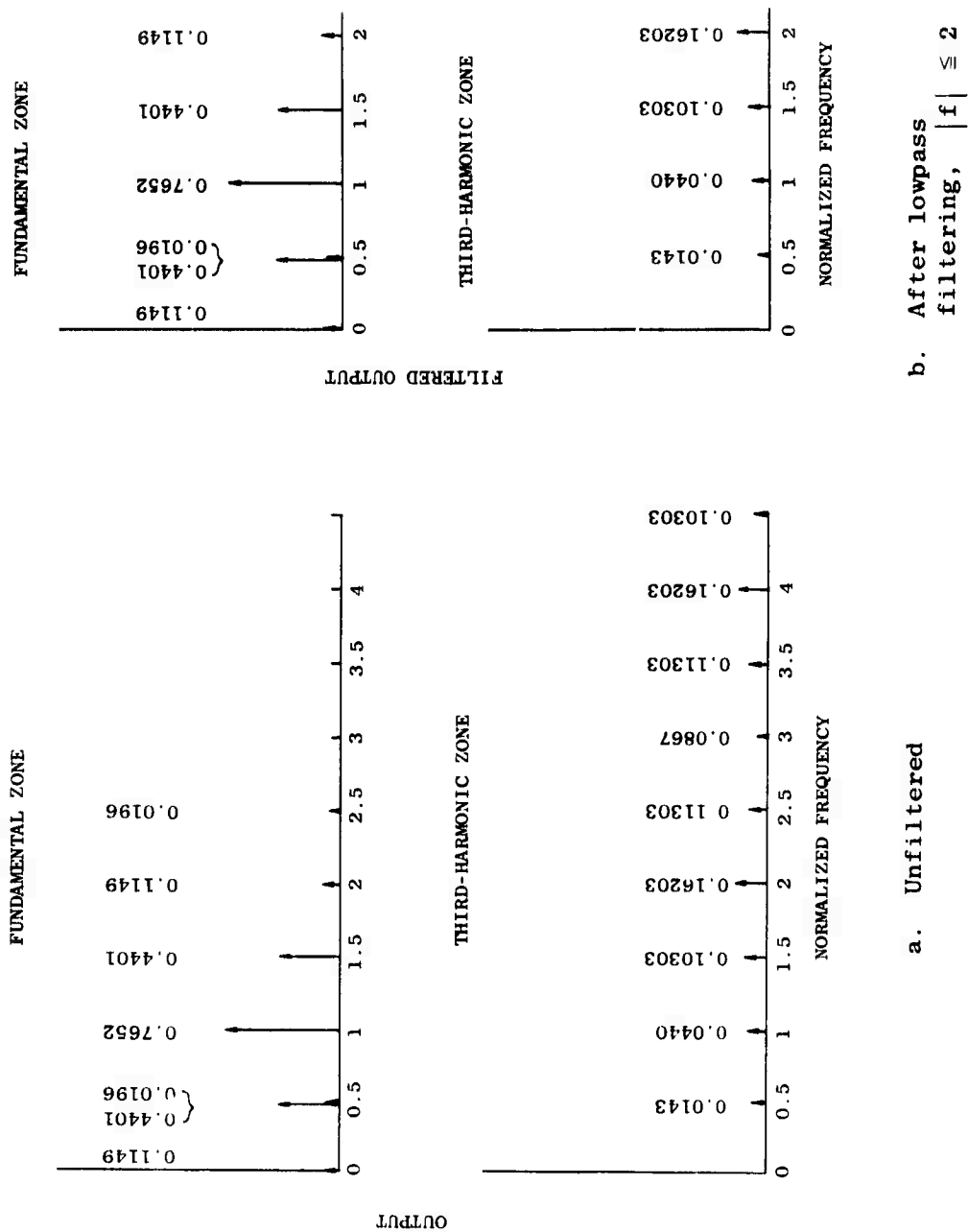


FIG. 10. SIGNIFICANT COMPONENTS OF THE FUNDAMENTAL AND THIRD-HARMONIC ZONES OF THE LIMITER OUTPUT.

switching between the normalized values of -1 to +1 with negligible rise and fall times. This assumption is usually valid in all practical systems.

2. FM Pulse Train

If any FM signal is an input to the limiter, then the train of pulses generated by placing an arbitrary but fixed pulse at every crossover or at every p^{th} crossover (where p is any positive integer) of the limited signal is termed an "FM pulse train."

An arbitrary but fixed pulse implies that the pulse shape, duration, amplitude, etc., may be arbitrarily chosen, but that once chosen, they remain invariant. The duration of the pulse is such that the successive pulses do not overlap at any instant.

It may be noted that if f_c is the nominal-carrier frequency and $\pm f$ is the peak frequency deviation, then the FM train with a pulse at every p^{th} crossover has a pulse repetition rate of $2f_c/p$ and a pulse deviation of $\pm 2f/p$.

Conceptually, and in practice as well, one of the best ways to generate or analyze an FM train is indicated in the block diagram of Fig. 11. The system analyzed is described in the following sections.

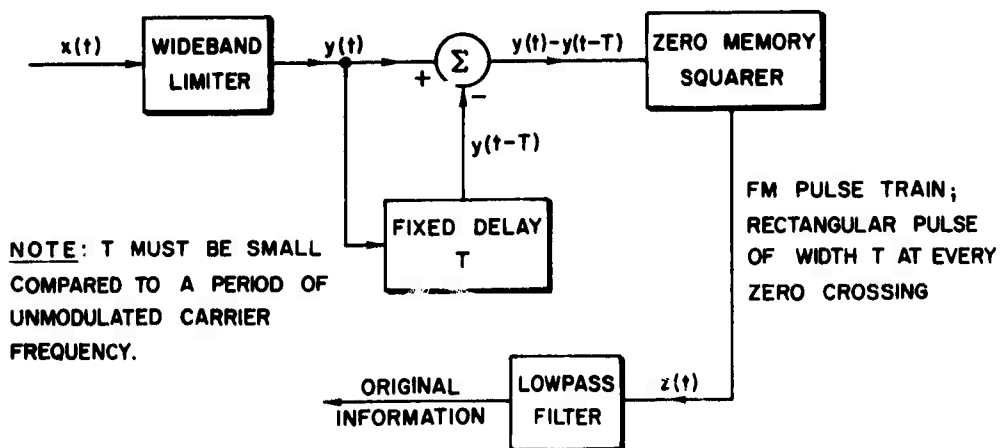


FIG. 11. BLOCK DIAGRAM OF THE SYSTEM ANALYZED.

a. Input Random Process, $x(t)$

Since the results for phase modulation can be written down directly from the results of frequency modulation, it will suffice to analyze the FM case only. Hence the input random process is assumed to be any signal of the form

$$x(t) = \cos [\omega_c t + \phi_c + \psi(t)] \quad (3.1)$$

where $x(t)$ is a process consisting of a sinusoidal waveform (of constant amplitude) whose phase is the sum of the random process $\psi(t)$ and the random variable ϕ_c . If $\psi(t)$ is the original information, then this process represents phase modulation. In the FM case, the derivative process $\dot{\psi}(t)$ represents the original information, that is,

$$\psi(t) \triangleq k_f \int^t g(\xi) d\xi \quad (3.2)$$

where $g(t)$ is the original information.

Define

$$\theta_1 \triangleq \omega_c t + \phi_c + k_f \int^t g(\xi) d\xi \quad (3.3)$$

and

$$\theta_2 \triangleq \omega_c (t - T) + \phi_c + k_f \int^{t-T} g(\xi) d\xi \quad (3.4)$$

and therefore

$$x(t) = \cos \theta_1$$

b. Square of the Limiter Output, $y^2(t)$

It can be shown that the output of the limiter $y(t)$ is

$$y(t) = \frac{4}{\pi} \sum_{\substack{m_1=1 \\ \text{odd}}}^{\infty} \frac{(-1)^{(m_1-1)/2} \cos m_1 \epsilon_1}{m_1} \quad (3.5)$$

Since the limiter has a large bandwidth, the rise and fall times are extremely small. The limiter output $y(t)$ switches from -1 to +1 and hence $y^2(t)$ has a constant value of 1 at all times. Also,

$$y^2(t) = \left(\frac{4}{\pi}\right)^2 \sum_{\substack{m_1=1 \\ \text{odd}}}^{\infty} \sum_{\substack{m_2=1 \\ \text{odd}}}^{\infty} \frac{(-1)^{(m_1+m_2-2)/2} \cos m_1 \epsilon_1 \cos m_2 \epsilon_2}{m_1 m_2} = 1 \quad (3.6)$$

c. Pulse Train, $z(t)$

The FM pulse train $z(t)$ consists of a train of positive-going (or negative-going) rectangular pulses of constant height and of constant width T placed at every crossover of the limiter output process $y(t)$. As shown in Fig. 11, the output of the limiter is delayed by a fixed amount, say T . The amount of delay T must be small compared with a period of the unmodulated carrier frequency. This delayed output is subtracted from $y(t)$ and the result passed through a zero memory squarer. This gives the pulse train $z(t)$, where

$$\begin{aligned} z(t) &= [y(t) - y(t - T)]^2 \\ &= y^2(t) + y^2(t - T) - 2y(t) y(t - T) \end{aligned}$$

Using Eq. (3.6) gives

$$z(t) = 2[1 - y(t) y(t - T)] \quad (3.7)$$

Hence the expression for $[y(t) y(t - T)]$ will give the expansion of the pulse train $z(t)$, the modifications being of a trivial nature.

C. EVALUATION OF $[y(t) y(t - T)]$

Equation (3.7) clearly shows the necessity of evaluating $[y(t) y(t - T)]$. With Eqs. (3.1) - (3.5), it can be shown that

$$y(t) y(t - T) = \left(\frac{4}{\pi}\right)^2 \sum_{\substack{m_1=1 \\ \text{odd}}}^{\infty} \sum_{\substack{m_2=1 \\ \text{odd}}}^{\infty} \frac{(-1)^{(m_1+m_2-2)/2} \cos m_1 \phi_1 \cos m_2 \phi_2}{m_1 m_2} \quad (3.8a)$$

$$= \left(\frac{4}{\pi}\right)^2 \frac{1}{2} \sum_{\substack{m_1=1 \\ \text{odd}}}^{\infty} \sum_{\substack{m_2=1 \\ \text{odd}}}^{\infty} \frac{(-1)^{(m_1+m_2-2)/2}}{m_1 m_2} [\cos (m_1 \phi_1 + m_2 \phi_2) + \cos (m_1 \phi_1 - m_2 \phi_2)] \quad (3.8b)$$

$$\triangleq C_0 + C_2 + C_4 + \dots + C_{2p} + \dots \quad (3.8c)$$

where C_{2p} represents the $2p^{\text{th}}$ spectral zone. From Eq. (3.8c) it is clear that the odd spectral zones do not exist.

1. Expression for the 0^{th} Spectral Zone, C_0

The 0^{th} spectral zone imposes the condition on the indices that

$$m_1 \pm m_2 = 0$$

with m_1 and m_2 each being positive odd integers. Hence the only contribution to this zone from the term $\cos (m_1 \theta_1 - m_2 \theta_2)$ is

$$\begin{aligned}
 c_0 &= \left(\frac{4}{\pi}\right)^2 \frac{1}{2} \sum_{\substack{m_1=1 \\ \text{odd}}}^{\infty} \frac{(-1)^{\frac{m_1-1}{2}}}{m_1^2} \cos m_1 (\theta_1 - \theta_2) \\
 &= \frac{1}{2} \left(\frac{4}{\pi}\right)^2 \sum_{\substack{m_1=1 \\ \text{odd}}}^{\infty} \frac{\cos m_1 (\theta_1 - \theta_2)}{m_1^2} \quad (3.9)
 \end{aligned}$$

From Bromwich [Ref. 28, p. 371],

$$\begin{aligned}
 \sum_{m_1=1}^{\infty} \frac{\cos m_1 X}{m_1^2} &= \lim_{a \rightarrow 0} \sum_{m_1=1}^{\infty} \frac{\cos m_1 X}{m_1^2 - a^2} \\
 &= \lim_{a \rightarrow 0} \frac{\pi \sin a \left(\frac{\pi}{2} - X\right)}{4a \cos \frac{\pi a}{2}} \\
 &= \frac{\pi}{4} \left(\frac{\pi}{2} - X\right) \quad 0 < X < \pi \quad (3.10)
 \end{aligned}$$

Hence

$$c_0 = 1 - \frac{2}{\pi} (\theta_1 - \theta_2) \quad 0 < (\theta_1 - \theta_2) < \pi \quad (3.11)$$

2. Expression for the $2p^{\text{th}}$ Spectral Zone, $p \neq 0$

The contributions to the $2p^{\text{th}}$ spectral zone can be written down as the total of the $\cos (m_1 \theta_1 + m_2 \theta_2)$ terms and the $\cos (m_1 \theta_1 - m_2 \theta_2)$ terms.

a. The $\cos (m_1 \theta_1 + m_2 \theta_2)$ Terms

These terms can be obtained by imposing the condition on the indices that

$$m_1 + m_2 = 2p \quad \text{or} \quad m_2 = 2p - m_1$$

Since m_1 and m_2 can each take on only positive odd integer values and since p can take all positive integers greater than zero, only those values of m_1 are acceptable which satisfy the condition

$$1 \leq m_1 \leq (2p - 1)$$

The application of this condition gives the subtotal C'_{2p} as

$$\begin{aligned} C'_{2p} &= \left(\frac{4}{\pi}\right)^2 \frac{1}{2} \sum_{\substack{m_1=1, \text{ odd} \\ m_2=(2p-m_1)}}^{\infty} \frac{(-1)^{(m_1+2p-m_1-2)/2}}{m_1(2p-m_1)} \cos [m_1 \theta_1 + (2p - m_1) \theta_2] \\ &= \left(\frac{4}{\pi}\right)^2 \frac{1}{2} \sum_{\substack{m_1=1 \\ \text{odd}}}^{2p-1} \frac{(-1)^p}{m_1(m_1 - 2p)} \cos [m_1(\theta_1 - \theta_2) + 2p\theta_2] \end{aligned} \quad (3.12)$$

b. The $\cos (m_1 \theta_1 - m_2 \theta_2)$ Terms

These terms can be obtained by using the condition

$$m_1 - m_2 = \pm 2p$$

Consider that $m_1 - m_2 = 2p$, that is, $m_2 = m_1 - 2p$. Also by noting that m_1 and m_2 can have all odd integer values greater than zero and the p can only have positive nonzero integer values, it can be shown that

$$(2p + 1) \leq m_1 \leq \infty$$

Therefore

$$\begin{aligned}
 C_{2p}'' &= \left(\frac{4}{\pi}\right)^2 \frac{1}{2} \sum_{\substack{m_1=2p+1, \text{ odd} \\ m_2 = (m_1 - 2p)}}^{\infty} \frac{(-1)^{[m_1 + (m_1 - 2p) - 2]/2}}{m_1(m_1 - 2p)} \cos [m_1 \theta_1 - (m_1 - 2p) \theta_2] \\
 &= \left(\frac{4}{\pi}\right)^2 \frac{1}{2} \sum_{\substack{m_1=2p+1 \\ \text{all odd}}}^{\infty} \frac{(-1)^p}{m_1(m_1 - 2p)} \cos [m_1(\theta_1 - \theta_2) + 2p\theta_2] \quad (3.13)
 \end{aligned}$$

Now consider $(m_1 - m_2) = -2p$, which implies $m_2 = (m_1 + 2p)$. Imposing the following conditions,

$$\left. \begin{aligned} p &\geq 0, & m_1 &\geq 1 \\ & & m_2 &\geq 1 \end{aligned} \right\} \text{all odd values}$$

gives $1 \leq m_1 < \infty$ which contributes C_{2p}''' as

$$\begin{aligned}
 C_{2p}''' &= \left(\frac{4}{\pi}\right)^2 \frac{1}{2} \sum_{\substack{m_1=1 \\ \text{odd}}}^{\infty} \frac{(-1)^{[m_1 + (m_1 + 2p) - 2]/2}}{m_1(m_1 + 2p)} \cos [m_1 \theta_1 - (m_1 + 2p) \theta_2] \\
 &= \left(\frac{4}{\pi}\right)^2 \frac{1}{2} \sum_{\substack{m_1=1 \\ \text{odd}}}^{\infty} \frac{(-1)^p}{m_1(m_1 + 2p)} \cos [m_1(\theta_1 - \theta_2) - 2p\theta_2] \quad (3.14a)
 \end{aligned}$$

$$\begin{aligned}
&= \left(\frac{4}{\pi}\right)^2 \frac{1}{2} \sum_{\substack{-m_1'=1 \\ \text{odd}}}^{\infty} \frac{(-1)^p}{(-m_1')[-(-m_1') + 2p]} \cos [-m_1'(\theta_1 - \theta_2) - 2p\theta_2] \\
&= \left(\frac{4}{\pi}\right)^2 \frac{1}{2} \sum_{\substack{m_1'=-1 \\ \text{odd}}}^{-\infty} \frac{(-1)^p}{m_1'(m_1' - 2p)} \cos [m_1'(\theta_1 - \theta_2) + 2p\theta_2] \quad (3.14b)
\end{aligned}$$

Hence

$$C_{2p} = C_{2p}' + C_{2p}'' + C_{2p}'''$$

$$= \left(\frac{4}{\pi}\right)^2 \frac{1}{2} \sum_{\substack{m_1=-\infty \\ \text{odd}}}^{\infty} \frac{(-1)^p}{m_1(m_1 - 2p)} \cos [m_1(\theta_1 - \theta_2) + 2p\theta_2] \quad (3.15a)$$

$$= \left(\frac{4}{\pi}\right)^2 \frac{1}{2} \frac{(-1)^p}{2p} \sum_{\substack{m_1=-\infty \\ \text{odd}}}^{\infty} \left\{ \frac{\cos [m_1(\theta_1 - \theta_2) + 2p\theta_2]}{m_1 - 2p} - \frac{\cos [m_1(\theta_1 - \theta_2) + 2p\theta_2]}{m_1} \right\} \quad (3.15b)$$

Further

$$\sum_{\substack{m_1=-\infty \\ \text{odd}}}^{\infty} \frac{\cos [(m_1(\theta_1 - \theta_2) + 2p\theta_2)]}{m_1 - 2p} = \sum_{\substack{m_1=-\infty \\ \text{odd}}}^{\infty} \frac{\cos [(m_1 - 2p)(\theta_1 - \theta_2) + 2p\theta_1]}{m_1 - 2p}$$

$$\begin{aligned}
&= \cos (2p\theta_1) \sum_{\substack{m_1=-\infty \\ \text{odd}}}^{\infty} \frac{\cos (m_1 - 2p)(\theta_1 - \theta_2)}{m_1 - 2p} \\
&\quad - \sin (2p\theta_1) \sum_{m_1=-\infty}^{\infty} \frac{\sin (m_1 - 2p)(\theta_1 - \theta_2)}{m_1 - 2p}
\end{aligned}$$

It is obvious that

$$\begin{aligned}
\sum_{\substack{n=-\infty, \text{ odd} \\ n \neq 0}}^{\infty} \frac{\cos n\theta}{n} &= \cos \theta (1 - 1) + \cos 3\theta \left(\frac{1}{3} - \frac{1}{3} \right) + \dots \\
&= 0
\end{aligned}$$

and, by Ref. 28,

$$\begin{aligned}
\sum_{\substack{n=-\infty \\ \text{odd}}}^{\infty} \frac{\sin n\theta}{n} &= 2 \sum_{\substack{n=1 \\ \text{odd}}}^{\infty} \frac{\sin n\theta}{n} \\
&= \frac{\pi}{2} \quad 0 < \theta < \pi \quad (3.16)
\end{aligned}$$

Hence

$$\sum_{\substack{m_1=-\infty \\ \text{odd}}}^{\infty} \frac{\cos [(m_1(\theta_1 - \theta_2) + 2p\theta_2)]}{m_1 - 2p} = -\frac{\pi}{2} \sin 2p\theta_1 \quad 0 < \theta < \pi \quad (3.17)$$

Also, proceeding on similar lines,

$$\sum_{\substack{m_1=-\infty \\ \text{odd}}}^{\infty} \frac{\cos [m_1(\theta_1 - \theta_2) + 2p\theta_2]}{m_1} = -\frac{\pi}{2} \sin 2p\theta_2 \quad 0 < \theta < \pi \quad (3.18)$$

Substituting the results of Eqs. (3.17) and (3.18) in Eq. (3.15b) gives

$$C_{2p} = \frac{4}{\pi} \frac{(-1)^p}{2p} [\sin 2p\theta_2 - \sin 2p\theta_1] \quad 0 < (\theta_1 - \theta_2) < \pi \quad (3.19a)$$

$$= -\frac{4}{\pi} \frac{(-1)^p \sin p(\theta_1 - \theta_2) \cos p(\theta_1 + \theta_2)}{p} \quad p \neq 0 \quad (3.19b)$$

Equations (3.8c), (3.11), and (3.19b) give

$$y(t) y(t-T) = \left[1 - \frac{2}{\pi} (\theta_1 - \theta_2)\right] - \frac{4}{\pi} \sum_{p=1}^{\infty} \frac{(-1)^p \sin p(\theta_1 - \theta_2) \cos p(\theta_1 + \theta_2)}{p} \quad 0 < (\theta_1 - \theta_2) < \pi \quad (3.20)$$

D. OUTPUT PROCESS

1. Expansion of $z(t)$

The expansion of the FM pulse train can be rewritten by substituting the result of Eq. (3.20) in Eq. (3.7), which gives

$$z(t) = \frac{4}{\pi} (\theta_1 - \theta_2) + 2 \frac{4}{\pi} \sum_{p=1}^{\infty} \frac{(-1)^p \sin p(\theta_1 - \theta_2) \cos p(\theta_1 + \theta_2)}{p}$$

$$\begin{aligned}
&= \frac{4}{\pi} \left[(\theta_1 - \theta_2) + 2 \sum_{p=1}^{\infty} \frac{(-1)^p \sin p(\theta_1 - \theta_2) \cos p(\theta_1 + \theta_2)}{p} \right] \\
&= \frac{4}{\pi} \left[(\theta_1 - \theta_2) + \sum_{p=1}^{\infty} (-1)^p \frac{\sin 2p\theta_1 - \sin 2p\theta_2}{p} \right] \quad (3.21)
\end{aligned}$$

where

θ_1 = total phase of the input process $x(t)$ at time t

θ_2 = total phase of the input process $x(t)$ at time $(t - T)$

It may be noted that

$$\theta_1 - \theta_2 = \omega_c T + [\psi(t) - \psi(t - T)] \quad (3.22)$$

$$\theta_1 + \theta_2 = 2 \left[\omega_c \left(t - \frac{T}{2} \right) + \phi_c \right] + [\psi(t) + \psi(t - T)] \quad (3.23)$$

where

$\psi(t)$ = information for phase modulation

$\dot{\psi}(t)$ = information for frequency modulation

It may be added that all the expressions derived above are valid for those cases where $x(t)$ is subjected to hard limiting with wideband limiters.

3. Autocorrelation Function

The derivation of the autocorrelation function, which is quite involved, is shown in Appendix B. The final result is contained in Eq. (B.19):

$$\begin{aligned}
R_z(t, t') = R_z(\tau) &= \left(\frac{4}{\pi} \right)^2 \left\{ \omega_c^2 \tau^2 + 2R_\psi(\tau) - [R_\psi(\tau + T) + R_\psi(\tau - T)] \right. \\
&+ \sum_{p=1}^{\infty} \sum_{n=-\infty}^{\infty} \frac{2J_n^2(2p\beta) \sin^2(2p\omega_c + n\omega_m) \frac{T}{2}}{p^2} \cos(2p\omega_c + n\omega_m)\tau \left. \right\} \quad (B.19)
\end{aligned}$$

3. Power Spectral Density

The expressions for the power spectral density of the output process are developed assuming rectangular pulse trains, arbitrarily shaped pulse trains, and power spectral density with no modulation.

a. Rectangular Pulses

In the case of rectangular pulse trains it is easily seen that

$$S_z(f) = \left(\frac{4}{\pi}\right)^2 \left\{ \omega_c^2 T^2 \delta(f) + 4 \sin^2 \frac{\omega T}{2} S_\psi(f) \right. \\ \left. + \sum_{p=1}^{\infty} \sum_{n=-\infty}^{\infty} \frac{2J_n^2(2p\beta)}{p^2} \sin^2 \left[(2p\omega_c + n\omega_m) \frac{T}{2} \right] \right. \\ \left. \cdot \frac{\delta[f - (2pf_c + nf_m)] + \delta[f + (2pf_c + nf_m)]}{2} \right\} \quad (3.24a)$$

$$= \left(\frac{4}{\pi}\right)^2 \left\{ \omega_c^2 T^2 \delta(f) + \left(2\beta^2 \sin^2 \frac{\omega T}{2}\right) \frac{\delta(f - f_m) + \delta(f + f_m)}{2} \right. \\ \left. + \sum_{p=1}^{\infty} \sum_{n=-\infty}^{\infty} \frac{2J_n^2(2p\beta) \sin^2 (2p\omega_c + n\omega_m) \frac{T}{2}}{p^2} \right. \\ \left. \cdot \frac{\delta[f - (2pf_c + nf_m)] + \delta[f + (2pf_c + nf_m)]}{2} \right\} \quad (3.24b)$$

b. Arbitrarily Shaped Pulses

For arbitrarily shaped pulse trains it is better to cast the expression for $S_z(f)$ into a more transparent form as follows:

$$\begin{aligned}
S_z(f) = & \left(\frac{4}{\pi}\right)^2 \left\{ (2f_c T)^2 \pi^2 \delta(f) + \left(2\beta^2 \frac{\sin^2 \pi f T}{\pi^2 f^2 T^2} \pi^2 f_m^2 T^2 \frac{\delta(f - f_m) + \delta(f + f_m)}{2} \right) \right. \\
& + \sum_{p=1}^{\infty} \sum_{n=-\infty}^{\infty} \left[\frac{2J_n^2(2p\beta)}{p^2} \frac{\sin^2 \pi f T}{\pi^2 f^2 T^2} \pi^2 (2pf_c + nf_m)^2 T^2 \right] \\
& \cdot \left. \frac{\delta[f - (2pf_c + nf_m)] + \delta[f + (2pf_c + nf_m)]}{2} \right\} \quad (3.25a)
\end{aligned}$$

$$\begin{aligned}
= & (4)^2 \left\{ (2f_c T)^2 \delta(f) + 2\beta^2 f_m^2 (T \operatorname{sinc} fT)^2 \frac{\delta(f - f_m) + \delta(f + f_m)}{2} \right. \\
& + \sum_{p=1}^{\infty} \sum_{n=-\infty}^{\infty} \frac{2(2pf_c + nf_m)^2 J_n^2(2p\beta)}{p^2} (T \operatorname{sinc} fT)^2 \\
& \cdot \left. \frac{\delta[f - (2pf_c + nf_m)] + \delta[f + (2pf_c + nf_m)]}{2} \right\} \quad (3.25b)
\end{aligned}$$

$$\begin{aligned}
= & (4)^2 (T \operatorname{sinc} fT)^2 \left\{ (2f_c)^2 \delta(f) + 2\beta^2 f_m^2 \frac{\delta(f - f_m) + \delta(f + f_m)}{2} \right. \\
& + \sum_{p=1}^{\infty} \sum_{n=-\infty}^{\infty} \frac{2(2pf_c + nf_m)^2 J_n^2(2p\beta)}{p^2} \\
& \cdot \left. \frac{\delta[f - (2pf_c + nf_m)] + \delta[f + (2pf_c + nf_m)]}{2} \right\} \quad (3.25c)
\end{aligned}$$

Also,

$$\mathcal{T}(t) \rightarrow T \text{ sinc } fT$$

Hence

$$R_z(\tau) = (4)^2 \mathcal{T}(t) * \mathcal{T}(t) * \left[(2f_c)^2 + 2\beta^2 f_m^2 \cos \omega_m \tau \right. \\ \left. + \sum_{p=1}^{\infty} \sum_{n=-\infty}^{\infty} \frac{2(2pf_c + nf_m)^2 J_n^2(2p\beta)}{p^2} \cos (2p\omega_c + n\omega_m) \tau \right] \quad (3.26)$$

Let $p(t)$ be any arbitrarily shaped pulse with $P(f)$ as its transform, and whose width is such that no overlapping of the successive pulses occurs. Then for such a pulse train $S_z(f)$ becomes

$$S_z(f) = (4)^2 [P(f)]^2 \left\{ (2f_c)^2 \delta(f) + 2\beta^2 f_m^2 \frac{\delta(f - f_m) + \delta(f + f_m)}{2} \right. \\ \left. + \sum_{p=1}^{\infty} \sum_{n=-\infty}^{\infty} \frac{2(2pf_c + nf_m)^2 J_n^2(2p\beta)}{p^2} \right. \\ \left. \cdot \frac{\delta[f - (2pf_c + nf_m)] + \delta[f + (2pf_c + nf_m)]}{2} \right\} \quad (3.27)$$

which gives the power spectral density for a train having arbitrarily but fixed shaped pulses at every zero crossing.

c. Output Spectra with No Modulation, $\beta = 0$

It can easily be shown from Eq. (3.25b) that the spectral power density with no modulation is

$$S_z(f) \Big|_{\text{no mod}} = (4)^2 (2f_c T)^2 \left[\delta(f) + \sum_{p=1}^{\infty} 2 \operatorname{sinc}^2(2pf_c T) \frac{\delta(f - 2pf_c) + \delta(f + 2pf_c)}{2} \right] \quad (3.28)$$

The above result can easily be derived in a simpler way by noting that the output with no modulation consists of a pulse train of random phase, the repetition rate being $2f_c$ and each pulse being $\sqrt{T}(t)$.

4. Output Voltage Vs Frequency Characteristic

It is convenient to express the sensitivity of the system shown in Fig. 11 in terms of the dc output voltage V_{dc} . From the expression for $S_z(f)$ it is easy to show that the rms voltage output at any frequency f_m is

$$\left(\frac{\beta V_{dc}}{\sqrt{2} f_c} \operatorname{sinc} f_m T \right) f_m \quad (3.29)$$

The linearity of such a system is therefore excellent.

E. INTERMODULATION PRODUCTS

From the power spectral density of $z(t)$, it is seen that the intermodulation products can be tagged by two indices, p and n . Hence $IM(p,n)$ would imply the following:

1. The intermodulation product $IM(p,n)$ belongs to the p^{th} spectral zone, p assuming all positive integer values.
2. The order of the intermodulation product n can have all integer values from $-\infty$ to $+\infty$.
3. The frequency of the intermodulation product $IM(p,n)$ is $(2pf_c + nf_m)$.
4. When the power in the wanted information is $[2\beta^2 \sin^2(\omega_m T/2)]$, then the power in the intermodulation product $IM(p,n)$ is

$$\frac{2J_n^2(2p\beta) \sin^2(2p\omega_c + n\omega_m) \frac{T}{2}}{p^2} \quad (3.30)$$

Since the power in $IM(p,n)$ is at a frequency of $(2pf_c + nf_m)$, it is possible for certain values of p and negative values of n that $IM(p,n)$ will lie in the wanted spectrum and hence cannot be filtered out. This intermodulation product appears in the output as an interference at a frequency which is not harmonically related to the information frequency and is therefore sometimes referred to as a "spurious output." In such cases, it is more convenient and meaningful to define a ratio, termed the "interference ratio," or $IR(p,n)$.

1. Interference Ratio, $IR(p,n)$

The interference ratio $IR(p,n)$ is defined as the power in $IM(p,n)$ when the power in the wanted information is normalized to 1. Hence

$$IR(p,n) = \text{interference at frequency } (2pf_c + nf_m)$$

$$= \left[\frac{J_n(2p\beta) \sin(2p\omega_c + n\omega_m) \frac{T}{2}}{p\beta \sin \frac{\omega_m T}{2}} \right]^2 \quad (3.31)$$

which can also be expressed in decibels.

Almost all existing wideband FM systems have operating parameters such that the values of $\omega_m T/2$ and $(2p\omega_c + n\omega_m)(T/2)$ are extremely small and the sines may be replaced by their arguments. This simplification gives

$$\begin{aligned} IR(p,n) &= \left\{ \frac{J_n(2p\beta)}{p\beta} \left[2p \left(\frac{f_c}{f_m} \right) + n \right] \right\}^2 \\ &= 20 \log_{10} \left[\frac{J_n(2p\beta)}{p\beta} \left| 2p \left(\frac{f_c}{f_m} \right) + n \right| \right] \text{ db} \end{aligned} \quad (3.32)$$

provided $f_m \neq 0$, $\beta \neq 0$, $p \in (1, \infty)$, and $n \neq 0$. It may be noted that n has a negative integer value in $IR(p, n)$.

The output $z(t)$ is filtered to recover the original information. Let the output of the final lowpass filter be B_{lp} . Hence all the intermodulation components which satisfy the condition $(2pf_c + nf_m) > B_{lp}$ are filterable and are therefore of no consequence as far as the overall system performance is concerned.

The strongest interference usually occurs for $p = 1$, and for the highest negative value of n which satisfies the condition

$$(2f_c + nf_m) \leq B_{lp}$$

that is, n is the integer which satisfies

$$n \leq - \frac{2f_c - B_{lp}}{f_m} \quad (3.33)$$

Since the highest negative value of n will occur for the highest in-band frequency, the worst interference will be for that n which is obtained when f_m is replaced by B_{lp} . The worst interference occurs at a frequency $(2f_c + nf_m)$, and has a ratio

$$IR(1, n) = 20 \log_{10} \left\{ \frac{J_{|n|}(2\beta)}{\beta} \left| 2 \left(\frac{f_c}{f_m} \right) + n \right| \right\} \text{ db} \quad (3.34)$$

where

$$\beta \triangleq \text{modulation index} = \pm \frac{\Delta f}{f_m}$$

Usually a $\pm k$ percent system implies that

$$\beta = \left| \pm \frac{k}{100} \frac{f_c}{f_m} \right| = 0.01 k \left(\frac{f_c}{f_m} \right) \quad (3.35)$$

Therefore

$$IR(1,n) = 20 \log_{10} \left\{ \frac{J_n \left[0.02 k \left(\frac{f_c}{f_m} \right) \right]}{0.01 k \left(\frac{f_c}{f_m} \right)} \left| 2 \left(\frac{f_c}{f_m} \right) + n \right| \right\} \text{ db} \quad (3.36)$$

It may be noted further that only the first-order interference components need be computed, since in most of the wideband FM systems the total contributions due to higher order components amount to less than 10 percent of the first-order term.

2. Pulse Shape and Interference Ratio

In computing the first-order effects, the interference ratio is calculated at a point where the equality $f_m = (2pf_c + nf_m)$ is satisfied. Hence the interference ratio is essentially unaffected by the shape of the pulses.

F. RESULTS

1. Theoretical and Experimental Results

In order to be able to select optimum system parameters, given a set of specifications, Eqs. (3.31) - (3.36) are used to compute: the interference ratios in Table 8 and the curves of Figs. 12 and 13.

1. Table 8 shows the interference ratio in decibels for various deviations. The information bandwidth is assumed to be 0 to 1, while the nominal-carrier frequency is set at 2. The interference ratios $IR(1,-3)$, $IR(1,-5)$, $IR(1,-7)$, and $IR(1,-11)$, which cause spurious outputs at frequencies 1, $2/3$, $1/2$, and $1/3$, are shown in the table.
2. The curves of Fig. 12 show graphically the relationship between the interference ratio and the deviation when the ratio of the nominal-carrier frequency to the information bandwidth is 2.
3. The curves of Fig. 13 show the behavior of spurious output vs carrier frequency at constant deviation

TABLE 8. INTERFERENCE RATIOS FOR VARIOUS
DEVIATIONS AND BANDWIDTHS

Normalized information bandwidth, $BW = 1$
Normalized nominal carrier frequency, $f_c/BW = 2$
(All values in minus decibels)

System Deviation ($\pm k\%$)	IR(1,-11) $f/BW = 1/3$	IR(1,-7) $f/BW = 1/2$	IR(1,-5) $f/BW = 2/3$	IR(1,-3) $f/BW = 1$
0	--	--	--	--
10	--	--	83.6	43.6
12.5	--	108.0	--	--
20	--	--	59.9	31.8
25	--	75.1	52.4	28.2
30	--	--	46.4	25.2
37.5	--	55.4	--	--
40	--	--	37.4	20.9
42	77.1	--	--	--
50	--	42.4	30.9	17.8

Note: Values not indicated were either negligibly small or not computed.

The experimentally measured results, which are also shown on Figs. 12 and 13, are in close agreement with the computed values. For example, spurious output measurements which were made for various deviations and for nominal-carrier and input frequencies were, in every case, within 2 db of the theoretically computed values. A typical example of these experimental measurements is reproduced below from data obtained on 6 November 1964:

	<u>Conditions</u>	
Nominal-carrier frequency:	900 kc	
Deviation:	± 40 percent	
Test gear:	FM multivibrator modulator	} prototypes
	FM pulse train demodulator	
	Spectrum analyzer	

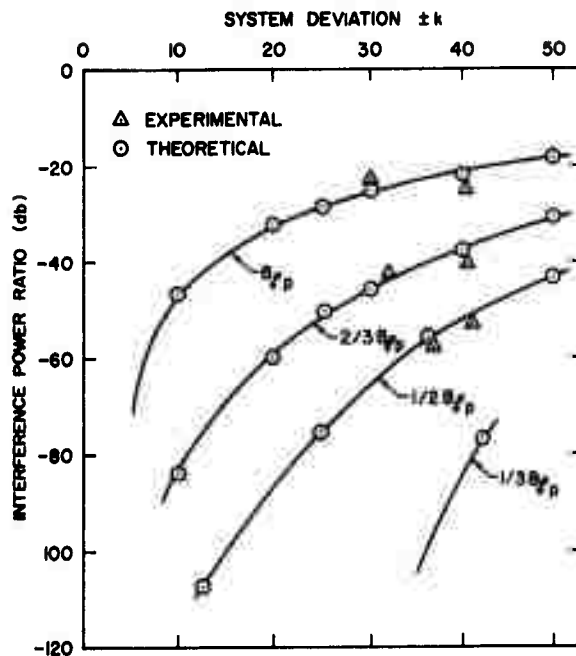


FIG. 12. CURVES SHOWING THE INTERFERENCE RATIO VS DEVIATION
FOR FIXED $f_c/f_{mm} = 2$.

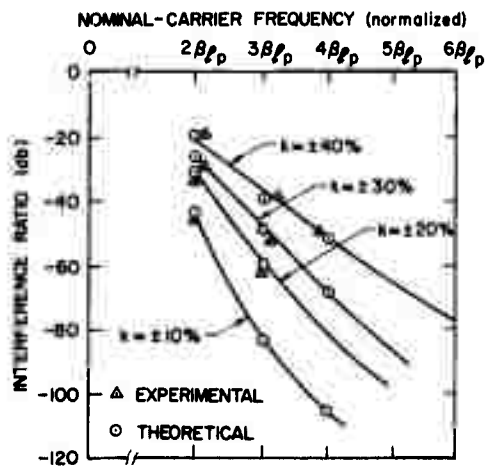


FIG. 13. CURVES SHOWING SPURIOUS OUTPUT BEHAVIOR VS CARRIER FREQUENCY
AT CONSTANT DEVIATION.

In Table 9 the computed and measured values for various $IR(p,n)$ are compared.

TABLE 9. COMPARISON OF COMPUTED AND MEASURED VALUES
FOR VARIOUS $IR(p,n)$

Input (kc)	$IR(p,n)$	Frequency (kc)	Computed Value (db)	Measured Value (db)
500	$IR(1,-3)$	300	-26.8	-29
	$IR(1,-4)$	200*	-37.8	-38
400	$IR(1,-5)$	200*	-52.5	-53
		550	-25.1	-27
250	$IR(1,-6)$	300	-42.4	-40
	$IR(1,-7)$	50	-71.3	--
	$IR(1,-14)$	400	-69.8	--
100	$IR(1,-15)$	300	--	--

*The input frequency had to be rocked slightly to distinguish between these two components.

The linearity and dc drift measurements were made on a typical FM pulse train demodulator with a nominal-carrier frequency of 900 kc and a deviation of ± 40 percent. The test gear used was a five-place digital voltmeter, a frequency counter, and a PM prototype demodulator. It was found that the maximum deviation from best straight line through zero was better than ± 0.1 percent and that the dc drift was better than 0.01 percent per degree Centigrade.

2. Advantages over Conventional Detectors

As far as low carriers and wide-deviation FM systems are concerned, this demodulation technique affords a better way to recover original information than the conventional FM discriminators or ratio detectors. The conventional detectors having a maximum deviation from linearity of less than ± 1 percent to over ± 50 percent of the center carrier frequency are extremely difficult to realize [see Ref. 24, p. 494; Ref. 29, p. 1104; and Ref. 30, p. 214].

1. FM pulse train demodulators represent at least an order-of-magnitude improvement in the linearity of frequency vs output voltage.
2. This technique results in a device having excellent dc drift characteristics.
3. It is cheaper and easy to implement.
4. Since the i-f filtering before detection leaves lower sidebands of the third harmonic of the carrier frequency (resulting in amplitude-modulation), the interference is too severe to permit satisfactory operation of the wideband FM discriminator.
5. The bandpass filter must be quite sharp and yet must have excellent envelope delay characteristics, which make the filtering after limiting (prior to detection) undesirable.

These advantages have resulted in the production of FM demodulators having the following specifications:

Center Carrier Frequency f_c	Deviation ($\pm\%$ of f_c)	Information Bandwidth	DC Drift ($\%/^{\circ}\text{C}$)	Maximum Deviation from Linearity ($\pm\%$)
1 to 1800 kc going up in octaves	40	dc to 1 Mc depending on f_c	0.01	0.1
7 Mc	30	dc to 5 Mc	--	1
15 Mc	30	dc to 10 Mc	--	1

IV. REDUCTION OF SPURIOUS OUTPUTS IN LOW-CARRIER, WIDE-DEVIATION, WIDE-BANDWIDTH FM SYSTEMS

A. FORMULATION OF THE PROBLEM

In many systems and practically all electromagnetic recorders, the channel has a bandwidth from a few hundred cycles to a few megacycles. In most cases the information has a bandwidth down to dc and therefore some form of a modulation scheme is required. The bandwidth of the channel, the modulation scheme, the tolerable strengths of extraneous outputs, and the signal-to-noise ratio are some of the factors which determine the bandwidth of the information that can be recorded. Frequently vestigial FM recording is used in order to record information of much larger bandwidths than is possible with conventional FM. The spectrum of a typical vestigial FM recording is shown in Fig. 14. A high modulation index is needed in order to obtain an acceptable signal-to-noise ratio.

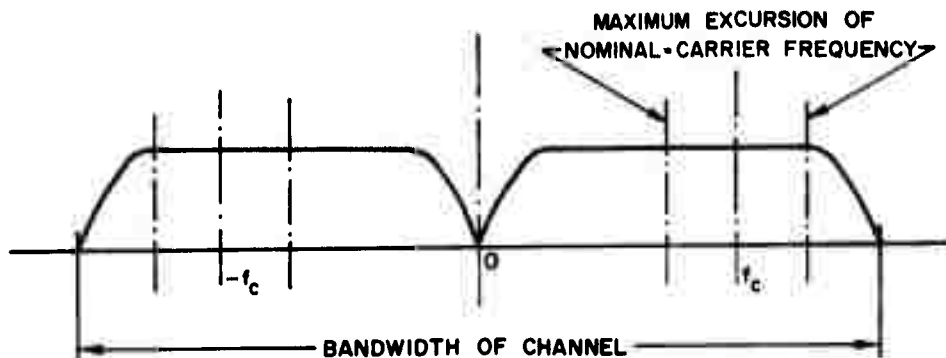


FIG. 14. TYPICAL BANDWIDTH OF CHANNEL, USING VESTIGIAL FREQUENCY MODULATIONS.

In the frequency-modulation case, some relations between the bandwidth of the modulated wave and the frequency deviation are given by Harris [Ref. 31] and Abramson [Ref. 32]. It is assumed in the analysis by Harris and also by Abramson that the input is a narrowband random process. This is shown in Fig. 15.

In the systems analyzed in this chapter, the narrowband assumption is not valid. A typical spectrum of a wideband system is shown in Fig. 17, p. 97. The spectrum centered at f_c partially overlaps the spectrum

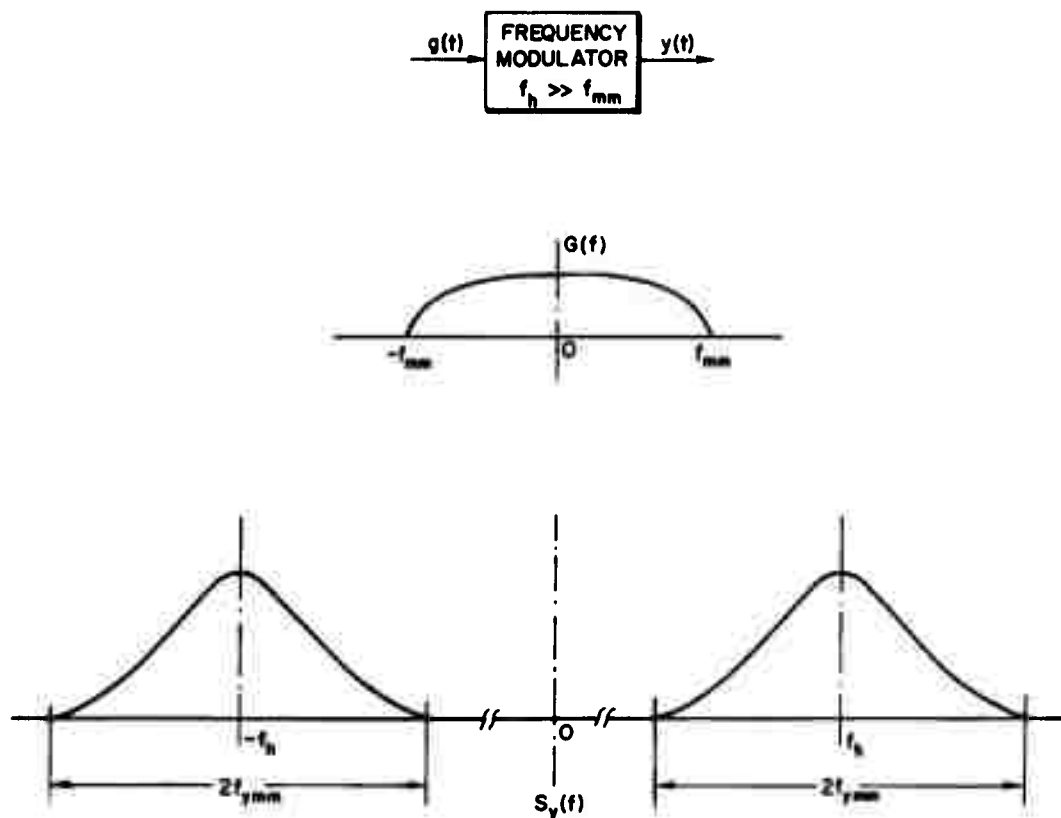


FIG. 15. MODULATED SPECTRUM WHEN $f_h \gg f_{y\text{mm}}$.

centered at $-f_c$ as a result of the nominal-carrier frequency being comparable to the information bandwidth. Because of this partial overlapping, "aliasing" occurs and outputs appear in the demodulated information which cannot be attributed to a harmonic distortion of the input information.[†] These so-called "suprious outputs" vanish under certain obvious conditions which, however, are unacceptable for the reasons indicated below:

<u>Condition</u>	<u>Reason for Nonacceptance</u>
1. Sufficiently high nominal-carrier frequency so that overlapping of the spectra does not occur.	Because of transducer electromagnetic limitations.
2. Reduced information bandwidth.	Makes the channel uneconomical.
3. Reduced deviation.	Results in rapid deterioration of signal-to-noise ratio.

[†]Wang and Wade [Ref. 33] discuss this phenomenon in connection with superregenerative parametric amplifiers.

The analysis of this chapter is devoted to finding acceptable methods, if there by any, for reducing the strengths of the spurious outputs.

B. ANALYSIS OF THE PROBLEM

It is assumed that the information $g(t)$ has a bandwidth $(-f_{mm}, f_{mm})$, where f_{mm} denotes the maximum frequency present in the information.

Define

$$\psi(t) = \int^t g(\xi) d\xi$$

If f_h is the nominal-carrier frequency of a frequency-modulated wave $y(t)$, and $f_h \gg f_{mm}$, then after normalizing the amplitude to 1,

$$y(t) = \cos [\omega_h t + \phi_h + \psi(t)] \quad (4.1)$$

It can easily be shown that if $\psi(t)$ is a gaussian random process with a finite mean square value [Ref. 32],

$$R_y(\tau) = \frac{1}{2} \exp [-R_\psi(0)] \exp [R_\psi(\tau)] \cos \omega_h \tau \quad (4.2)$$

where $R_y(\tau)$ is the autocorrelation function of $y(t)$. The power spectral density of $y(t)$ is (see Fig. 15),

$$S_y(f) = \frac{1}{4} \exp [-R_\psi(0)] \mathfrak{F}\{\exp [R_\psi(\tau)]\} * [\delta(f - f_h) + \delta(f + f_h)] \quad (4.3)$$

where the Fourier transform of $R_\psi(\tau)$ is

$$\mathfrak{F}\{\exp [R_\psi(\tau)]\} = \int_{-\infty}^{\infty} \exp [R_\psi(\tau)] \exp (-i\omega\tau) d\tau$$

Although the information $g(t)$ is bandlimited to $f \leq |f_{mm}|$, it can be seen from Fig. 15 that the bandwidth of $S_y(f)$ is given by expression (4.3). In the following discussion it is assumed that $S_y(f)$ exists in

a band $2f_{ymm}$ centered at $\pm f_h$ and is practically zero elsewhere. Usually $f_{ymm} > f_{mm}$.

If the conditions in Fig. 15 exist, that is, $f_h \gg f_{ymm}$, then demodulation by any standard technique will result in the recovery of the information $g(t)$ without spurious outputs.

If $y(t)$ is now translated down to another frequency, say f'_h is greater than f_{ymm} as shown in Fig. 16, the original message can still be recovered without any spurious outputs. This is evident from the fact that if one reverts back to the original carrier frequency f_h by translation, as shown in Fig. 16, one obtains $y(t)$ exactly. Hence the whole process is "reversible."

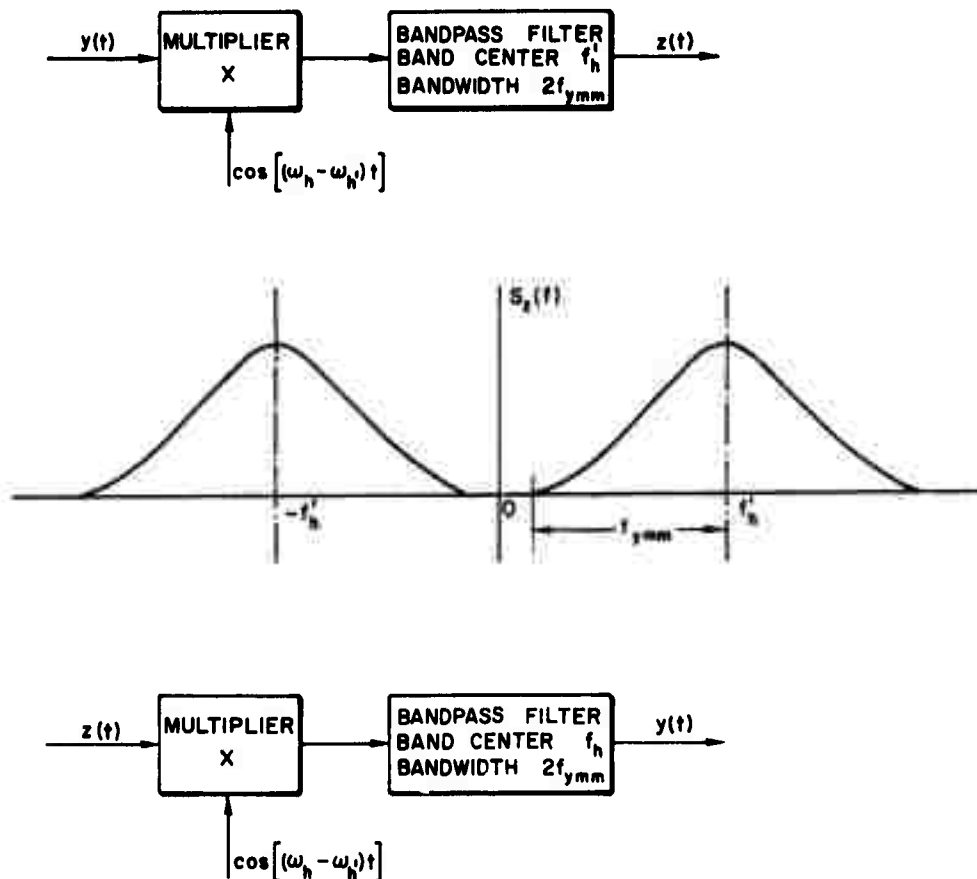


FIG. 16. MODULATED SPECTRUM WHEN $f'_h \geq f_{ymm}$.

The case where $y(t)$ is translated down to a carrier frequency f_c less than f_{ymm} is shown in Fig. 17. This condition is usually prevalent in wideband FM recorders. Partial overlapping of the positive and negative spectra occurs and spurious outputs result. The shape of the spectrum of $S_z(f)$ is different from $S_y(f)$; and given such a $z(t)$, one cannot revert back to $y(t)$ by translation. The process is therefore "irreversible."

C. INTERRELATIONSHIPS OF f_c , f_{ymm} , and f_{lp} FOR FILTERABLE SPURIOUS OUTPUTS

If one demodulates $z(t)$ and passes the output through a lowpass filter to recover the message $g(t)$, a portion of the spurious outputs

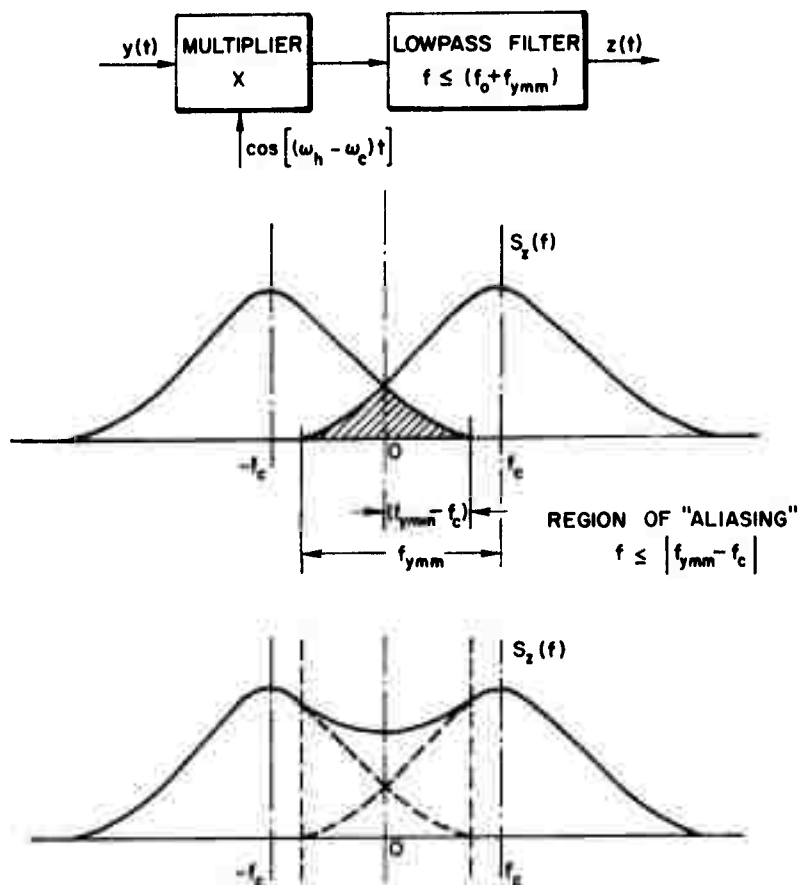


FIG. 17. MODULATED SPECTRUM WHEN $f_c < f_{ymm}$ (SHOWS ALIASING). As a result of aliasing, $y(t)$ cannot be recovered from $z(t)$. The dotted lines in the lower figure indicate the shape of $S_y(f)$ and all $S_z(f)$ for $f_c > f_{ymm}$.

generated due to the aliasing phenomenon is filterable (Fig. 18). To be more specific, if the lowpass filter has a bandwidth of f_{lp} , then from Fig. 17 and

$$f_{lp} \leq [f_c - (f_{ymm} - f_c)] = 2f_c - f_{ymm} \quad (4.4)$$

minimum distortion occurs in $g(t)$ since the spurious outputs are filterable. Hence if $g(t)$ is bandlimited to frequencies less than $(2f_c - f_{ymm})$, good demodulation is possible.



FIG. 18. LOWPASS FILTERED OUTPUT.

In the frequency-modulation case, the spectrum theoretically has an infinite bandwidth. In all practical systems, however, $S_z(f)$ is significant over a small frequency interval. All components of $S_z(f)$ outside this interval may be very much below the signal-to-noise ratio of the system, which sets a criterion in estimating f_{ymm} from Eq. (4.4). To illustrate the application of the result, consider a simple case of monotone modulation. If the modulation index β for the maximum frequency of information f_{mm} is such that only p lower sidebands are significantly different from zero, then for the case of no spurious outputs in the filtered output due to aliasing,

$$f_{mm} \leq (2f_c - pf_{mm}), \quad \text{that is,} \quad f_{mm} = \left(\frac{2}{p+1} \right) f_c \quad (4.5)$$

D. REDUCTION OF SPURIOUS OUTPUTS

Since the bandwidth of the channel is less than f_{ymm} , the best one can do for minimization of spurious components is to make $S_z(f)$ in Fig. 17 coincide with $S_y(f)$ throughout the entire bandwidth of the channel, as shown by the dotted lines in Fig. 17. This approximation can be done by appropriate filtering or phasing techniques.

1. Filtering Technique

It is clear that the filtering should be performed when the conditions in Fig. 15 prevail. The bandpass filter must have an approximately constant group delay characteristic close to the first null. The slope of the filter depends upon f_{mm} , f_c , f_h , the modulation index β , the signal-to-noise ratio of the system, and the rate of cutoff of the final lowpass filter at the output of the receiver. With a set of Bessel function tables and a reasonably low value of f_h , the system parameters can be chosen so as to result in filters that are realistic and inexpensive. The complete system which implements the filtering technique is shown in Figs. 19 and 20.

2. Phasing Technique

The block diagram which implements the phasing technique is shown in Fig. 21. The modulator of Fig. 21a and the demodulator of Fig. 21b have both resulted from the following theoretical analysis:

$$\begin{aligned} y(t) &= \cos [\omega_c t + \phi_c + \psi(t)] \\ &= \cos (\omega_c t + \phi_c) [\cos \psi(t)] - \sin (\omega_c t + \phi_c) [\sin \psi(t)] \end{aligned}$$

Taking Fourier transforms,

$$\begin{aligned} Y(f) &= \left[\frac{e^{i\phi_c}}{2} \delta(f - f_c) + \frac{e^{-i\phi_c}}{2} \delta(f + f_c) \right] * \mathcal{F}\{\cos \psi(t)\} \\ &\quad - \left[\frac{e^{i\phi_c}}{2i} \delta(f - f_c) - \frac{e^{-i\phi_c}}{2i} \delta(f + f_c) \right] * \mathcal{F}\{\sin \psi(t)\} \end{aligned} \quad (4.6)$$

Define

$$C_{-f_c} = \mathcal{F}\{\cos \psi(t)\} * \delta(f + f_c) \quad (4.7a)$$

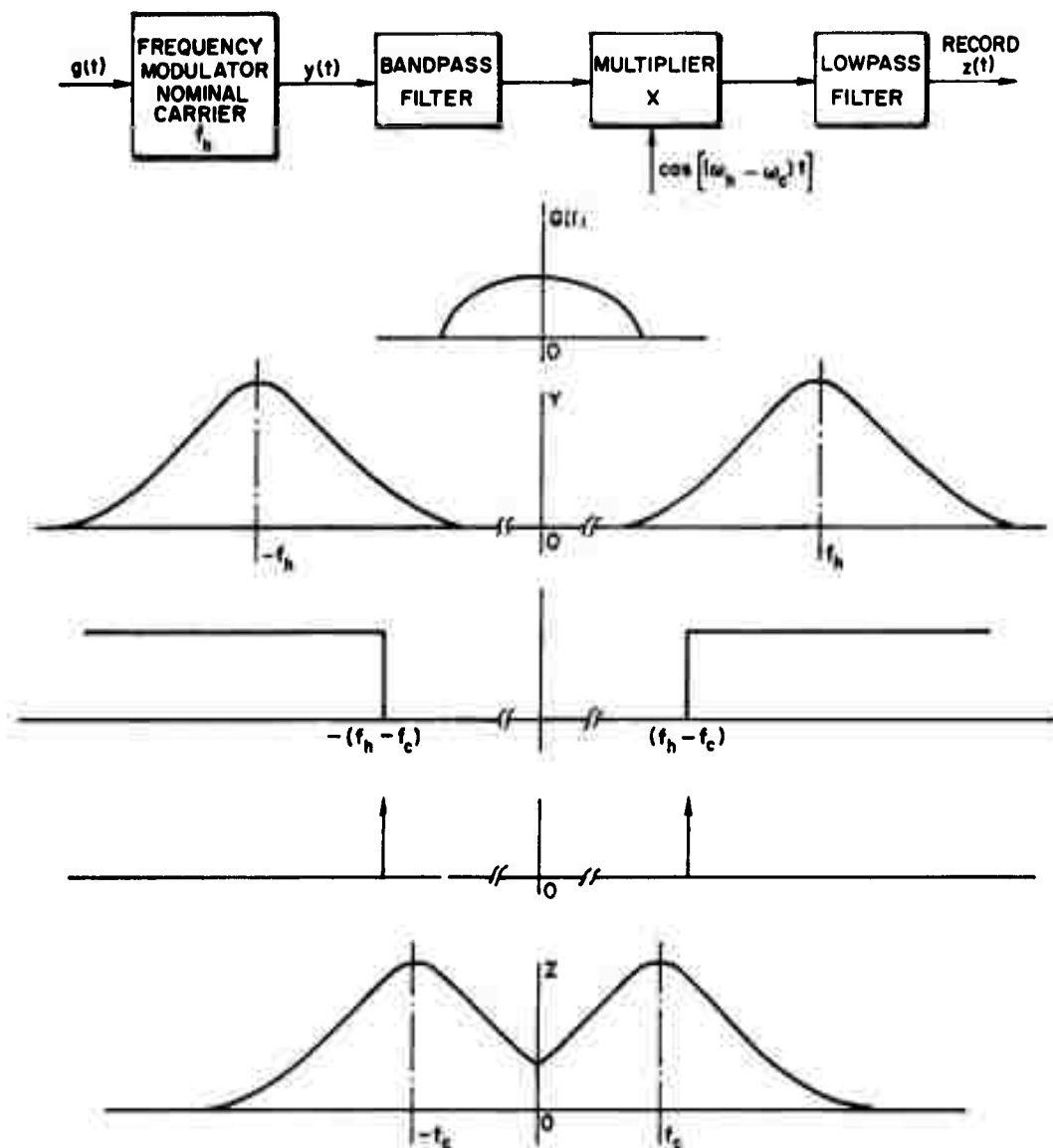


FIG. 19. BLOCK DIAGRAM AND SPECTRA OF THE PROPOSED SYSTEM USING THE FILTERING TECHNIQUE.

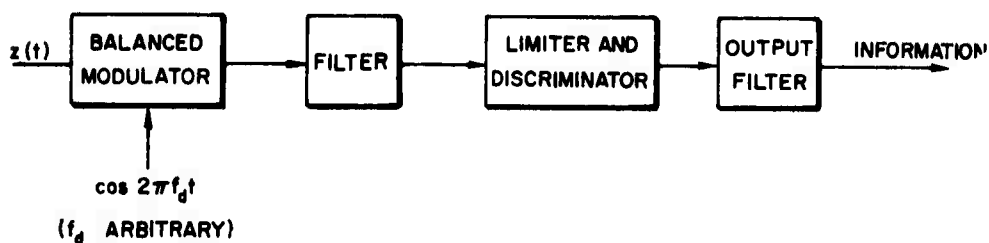
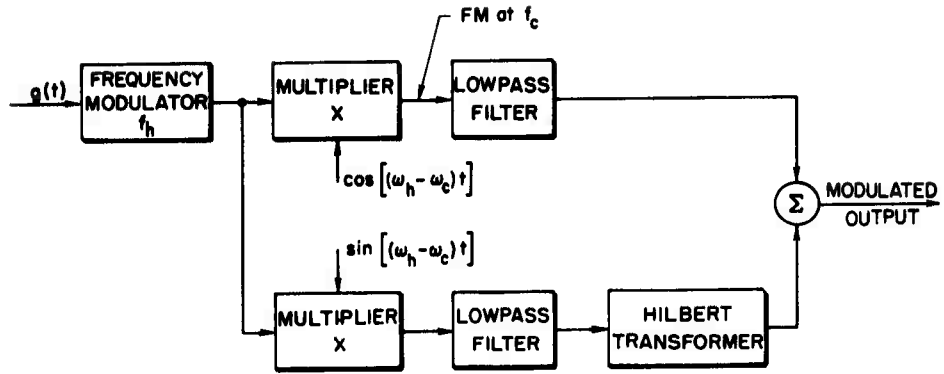
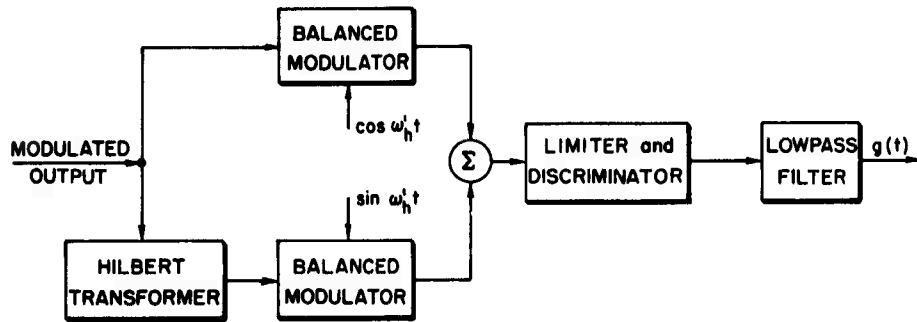


FIG. 20. DEMODULATION PROCESS USING THE FILTERING TECHNIQUE.



a. Modulation



b. Demodulation

FIG. 21. BLOCK DIAGRAMS OF THE PHASING TECHNIQUE.

$$C_{+f_c} = \mathfrak{F}\{\cos \psi(t)\} * \delta(f - f_c) \quad (4.7b)$$

$$S_{-f_c} = \mathfrak{F}\{\sin \psi(t)\} * \delta(f + f_c) \quad (4.7c)$$

$$S_{+f_c} = \mathfrak{F}\{\sin \psi(t)\} * \delta(f - f_c) \quad (4.7d)$$

$$[F(f)]_+ \triangleq F(f) \frac{1}{2} [1 + \operatorname{sgn} f]$$

$$= F(f) \quad \text{for all } f < 0$$

$$= 0 \quad \text{for all } f > 0 \quad (4.8)$$

$$\begin{aligned}
[F(f)]_- &\triangleq F(f) \frac{1}{2} [1 - \operatorname{sgn} f] \\
&= F(f) \quad \text{for all } f < 0 \\
&= 0 \quad \text{for all } f > 0
\end{aligned} \tag{4.9}$$

Hence

$$Y(f) = \frac{e^{i\phi_c}}{2} C_{+f_c} + \frac{e^{-i\phi_c}}{2} C_{-f_c} - \frac{e^{i\phi_c}}{2i} S_{+f_c} + \frac{e^{i\phi_c}}{2i} S_{-f_c} \tag{4.10}$$

also define

$$r(t) = \sin [\omega_c t + \phi_c + \psi(t)] \tag{4.11}$$

$$\begin{aligned}
R(f) &= \left[\delta(f - f_c) \frac{e^{i\phi_c}}{2i} - \frac{e^{-i\phi_c}}{2i} \delta(f + f_c) \right] * \mathfrak{F}\{\cos \psi(t)\} \\
&\quad + \left[\frac{e^{i\phi_c}}{2} \delta(f - f_c) + \frac{e^{-i\phi_c}}{2} \delta(f + f_c) \right] * \mathfrak{F}\{\sin \psi(t)\} \\
&= \frac{e^{i\phi_c}}{2i} C_{+f_c} - \frac{e^{-i\phi_c}}{2i} C_{-f_c} + \frac{e^{i\phi_c}}{2} S_{+f_c} + \frac{e^{-i\phi_c}}{2} S_{-f_c}
\end{aligned} \tag{4.12}$$

Defining the Hilbert transform of $x(t)$ as $\hat{x}(t)$,

$$\hat{x}(t) = \text{Cauchy's principal value of } \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{x(s)}{t - s} ds$$

$$\begin{aligned}\hat{R}(f) = & -\frac{e^{i\phi_c}}{2} \left(C_{+f_c} \right)_+ + \frac{e^{i\phi_c}}{2} \left(C_{+f_c} \right)_- + \frac{e^{-i\phi_c}}{2} \left(C_{-f_c} \right)_+ - \frac{e^{-i\phi_c}}{2} \left(C_{-f_c} \right)_- \\ & - \frac{ie^{i\phi_c}}{2} \left(S_{+f_c} \right)_+ + \frac{ie^{i\phi_c}}{2} \left(S_{+f_c} \right)_- - \frac{ie^{-i\phi_c}}{2} \left(S_{-f_c} \right)_+ + \frac{ie^{-i\phi_c}}{2} \left(S_{-f_c} \right)_-\end{aligned}\quad (4.13)$$

Using the relation

$$F(f) = [F(f)]_+ + [F(f)]_- \quad (4.14)$$

and subtracting $\hat{R}(f)$ from $Y(f)$ gives

$$\frac{Y(f) - \hat{R}(f)}{2} = \frac{e^{i\phi_c}}{2} \left(C_{+f_c} \right)_+ + \frac{e^{-i\phi_c}}{2} \left(C_{-f_c} \right)_- - \frac{ie^{i\phi_c}}{2} \left(S_{+f_c} \right)_+ + \frac{ie^{-i\phi_c}}{2} \left(S_{-f_c} \right)_- \quad (4.15)$$

A comparison of the above equation with Eq. (4.10) for $Y(f)$ shows that the wanted output is precisely $Y(f) - \hat{R}(f)$ since the interfering portions of the positive and negative spectra nullified each other. The block diagram of the modulator based on this analysis is shown in Fig. 21a. To recover the message, the systems shown in Fig. 19 or Fig. 21b can be used.

It may be added that Hilbert transformers can be approximated to a desired accuracy over a wide frequency interval by several methods, e.g., an active network, a passive network, or a tapped-delay-line realization. Detailed explanations of these methods are contained in Refs. 34 through 39. The phasing method with modifications can be used to develop complete systems using SSB-FM, SSB-PM or SSB-Hybrid modulation. Some experimental results on SSB-FM are contained in Ref. 39.

The analysis presented in this chapter has resulted in techniques which suppress the spurious outputs that result from the interaction of the positive and negative spectra when the nominal-carrier frequency is

sufficiently lowered. Block diagrams of the proposed systems are presented in Figs. 20, 21a, and 21b. Although the solutions are equally effective, for reasons of economy the two systems shown in Figs. 20 and 21a would be preferable.

V. CONCLUDING DISCUSSION

An analysis, together with experimental verification of the theoretical results, has been presented which allows the computation of the strength in any output component and the "distortion" (i.e., the sum of the power in all the intermodulation components in the first-harmonic band) when many signals in noise are subjected to hard limiting. Typical situations in which these results have proved to be most useful are as follows:

1. When several modulated or unmodulated carriers are subjected to hard limiting--a problem of common occurrence in a satellite communication system or in radar problems involving multiple echoes.
2. When a frequency-modulated signal, after being passed through a channel having arbitrary output vs frequency characteristics, is subjected to hard limiting.
3. This study may be used by various organization, such as NASA's Inter-Range Instrumentation Group (IRIG), for drawing up specifications for various frequency-modulated instrumentation systems.

For the general case, expressions are developed which give the signal-suppression effect, the output signal power distribution, and the power in the various crossproducts due to hard limiting. A few results that are well known for the cases of one signal in noise and two signals in noise are shown to be valid for the general case.

It is also shown that the behavior on hard limiting of the many-signals-in-noise case approaches that of the one-strong-signal-in-noise case if the strong signal power is much greater than the sum of the power in all the remaining weak signals and noise. It should be stressed that the power in each of the weak signals or noise need not be equal for the above approximation to be valid. Further, the case of hard limiting of many signals in noise is reducible to that of two signals in noise if the power in each of the two signals is much greater than the sum of the power in the remaining weak signals and noise. The approximation is valid for any arbitrary weak signal and noise power distribution since it is the

sum of the power in them that is of consequence. Also the case of p signals in noise can be approximated as the case of $(p + 1)$ signals without noise, as far as the behavior due to limiting is concerned. Further, the output power in the various components of the limited signal is relatively independent of the input strong-signal-to-noise ratios, for ratios larger than 10 db. For the case of p equal signals (with and without noise), the results of hard limiting for a large range of p are presented.

Some of the contributions of this research are listed below:

1. A means of predicting the approximate behavior on hard limiting of many signals and noise has been developed.
2. Approximate expressions for the power in the various output signal and crossproduct components for any p , derived analytically are given. These expressions give results which are very close to those obtained through extensive digital computer computations, the error in most cases being negligible.
3. The judicious selection is permitted of frequency spacings of several modulated carriers so as to minimize the distortion over any frequency band due to hard limiting.
4. The strength of the interfering components present at the output of an FM system, accomplishing demodulation by hard limiting followed by integrating an FM pulse train with arbitrary pulses at every p^{th} crossover, can be easily evaluated by the results contained in this report.
5. The phenomenon of "aliasing," is analyzed, the optimum selection of operating parameters is permitted, and systems are proposed which minimize such distortion.

APPENDIX A. USEFUL INTEGRALS OF BESSEL FUNCTIONS

Recently much work has been done in the evaluation of integrals involving Bessel functions and products of Bessel functions. Some of the important references and pertinent equations are given below.

$$\begin{aligned}
 \int_0^{\infty} \exp(-p^2 t^2) t^{\mu-1} J_{\nu}(at) dt &= \frac{(a/2p)^{\nu} \Gamma\left(\frac{\nu+\mu}{2}\right)}{2p^{\mu} \Gamma(\nu+1)} \\
 &\cdot {}_1F_1\left(\frac{\mu+\nu}{2}; \nu+1; -\frac{a^2}{4p^2}\right) \\
 &= \frac{\exp(-a^2/4p) (a/2p)^{\nu} \Gamma\left(\frac{\mu+\nu}{2}\right)}{2p^{\mu} \Gamma(\nu+1)} \\
 &\cdot {}_1F_1\left(\frac{\nu-\mu}{2}+1; \nu+1; \frac{a^2}{4p^2}\right)
 \end{aligned}
 \tag{A.1}$$

with the conditions

$$R(\mu + \nu) > 0, \quad R(p^2) > 0$$

For tables of the confluent hypergeometric function, ${}_1F_1$, refer to Slater [Ref. 12].

$$\begin{aligned}
 \int_0^{\infty} t^{\lambda} \exp(-p^2 t^2) J_{\mu}(at) J_{\nu}(bt) dt &= \frac{\frac{1}{2} (a/2p)^{\mu} (b/2p)^{\nu} \Gamma\left(\frac{\mu+\nu+\lambda+1}{2}\right)}{p^{\lambda+1} \Gamma(\mu+1) \Gamma(\nu+1)} \\
 &\cdot \sum_{k=0}^{\infty} \frac{(-)^k (a/2p)^{2k} \left(\frac{\mu+\nu+\lambda+1}{2}\right)_k}{k! (\mu+1)_k}
 \end{aligned}$$

$$\cdot {}_2F_1 \left(-k, -\mu-k; \nu+1; \frac{b^2}{a^2} \right) \quad (\text{A.2})$$

with the conditions

$$R(\mu + \nu + \lambda) > -1, \quad R(p^2) > 0$$

where $(\alpha)_n$ is the generalized factorial function defined as [Ref. 13]

$$\begin{aligned} (\alpha)_n &\triangleq \prod_{k=1}^n (\alpha + k - 1) \\ &= \alpha(\alpha+1)(\alpha+2) \dots (\alpha+n-1) \quad n \geq 1 \end{aligned} \quad (\text{A.3})$$

and

$$(\alpha)_0 = 1 \quad \alpha \neq 0$$

The following Weber-Schafheitlin type integrals are treated in great detail in many references [see for example Watson, Ref. 11]

$$\begin{aligned} \int_0^\infty t^{-\lambda} J_\mu(at) J_\nu(bt) dt &= \frac{(b/a)^\nu (a/2)^{\lambda-1} \Gamma\left(\frac{\mu+\nu-\lambda+1}{2}\right)}{2\Gamma(\nu+1) \Gamma\left(\frac{\mu-\nu-\lambda+1}{2}\right)} \\ &\cdot {}_2F_1 \left(\frac{\mu+\nu-\lambda+1}{2}, \frac{\nu-\mu-\lambda+1}{2}; \nu+1; \frac{b^2}{a^2} \right) \end{aligned} \quad (\text{A.4a})$$

with the conditions

$$R(\mu + \nu - \lambda + 1) > 0, \quad R(\lambda) > -1, \quad 0 < b < a$$

$$\int_0^\infty t^{-\lambda} J_\mu(at) J_\nu(bt) dt = \frac{(a/b)^\nu (b/2)^{\lambda-1} \Gamma\left(\frac{\mu + \nu - \lambda + 1}{2}\right)}{2\Gamma(\nu + 1) \Gamma\left(\frac{\nu - \mu + \lambda + 1}{2}\right)} \cdot {}_2F_1\left(\frac{\mu + \nu - \lambda + 1}{2}, \frac{\mu - \nu - \lambda + 1}{2}; \mu + 1; \frac{a^2}{b^2}\right) \quad (\text{A.4b})$$

with the conditions

$$R(\mu + \nu - \lambda + 1) > 0, \quad R(\lambda) > -1, \quad 0 < a < b$$

Note that (A.4a) and (A.4b) are not continuous at $a = b$ and that

$$\int_0^\infty t^{-\lambda} J_\mu(at) J_\nu(bt) dt = \frac{(a/2)^{\lambda-1} \Gamma(\lambda) \Gamma\left(\frac{\mu + \nu - \lambda + 1}{2}\right)}{2\Gamma\left(\frac{\nu - \mu + \lambda + 1}{2}\right) \Gamma\left(\frac{\nu + \mu + \lambda + 1}{2}\right) \Gamma\left(\frac{\mu - \nu + \lambda + 1}{2}\right)} \quad (\text{A.5})$$

with the conditions

$$R(\mu + \nu + 1) > R(\lambda) > 0, \quad a > 0$$

For integrals involving the product of more than two Bessel functions, it is possible to expand the other Bessel functions in powers of t , and to use expressions (A.4b) or (A.1). It was found best not to use the above expressions for the general case as it is easier and more convenient to carry out the integrations numerically on a digital computer.

APPENDIX B. DERIVATION OF AUTOCORRELATION
FUNCTION OF THE OUTPUT $z(t)$

It is convenient to define

$$\xi_T(t) = [\psi(t) - \psi(t - \tau)] \quad (\text{B.1})$$

Therefore, $(\theta_1 - \theta_2) = \omega_c T + \xi_T(t)$. The autocorrelation function of the output $z(t)$ is defined as

$$R_z(t, t') = \left(\frac{4}{\pi}\right)^2 E \left\{ \left[\omega_c T + \xi_T(t) + \sum_{p=1}^{\infty} \frac{(-1)^p}{p} (\sin 2p\theta_{1t} - \sin 2p\theta_{2t}) \right] \right. \\ \left. \cdot \left[\omega_c T + \xi_T(t') + \sum_{q=1}^{\infty} \frac{(-1)^q}{q} (\sin 2q\theta_{1t'} - \sin 2q\theta_{2t'}) \right] \right\} \quad (\text{B.2a})$$

where E denotes the expectation and θ_{1t} denotes the value of θ_1 at the instant t . Hence

$$R_z(t, t') = \left(\frac{4}{\pi}\right)^2 \left\{ E \left[\xi_T(t) \cdot \xi_T(t') \right] + E \left[\sum_{p=1}^{\infty} \sum_{q=1}^{\infty} \frac{(-1)^{p+q}}{pq} (\sin 2p\theta_{1t} - \sin 2p\theta_{2t}) \right. \right. \\ \left. \left. \cdot (\sin 2q\theta_{1t'} - \sin 2q\theta_{2t'}) \right] \right\} \\ + E \left\{ \xi_T(t) \cdot \xi_T(t') + \sum_{p=1}^{\infty} \frac{(-1)^p}{p} (\sin 2p\theta_{1t} - \sin 2p\theta_{2t}) \right. \\ \left. \cdot \sum_{q=1}^{\infty} \frac{(-1)^q}{q} (\sin 2q\theta_{1t'} - \sin 2q\theta_{2t'}) \right\} \\ + E \left\{ \sum_{p=1}^{\infty} \frac{(-1)^p}{p} \xi_T(t) (\sin 2p\theta_{1t'} - \sin 2p\theta_{2t'}) \right\} + E \left\{ \sum_{q=1}^{\infty} \frac{(-1)^q}{q} \xi_T(t') (\sin 2q\theta_{1t} - \sin 2q\theta_{2t}) \right\} \quad (\text{B.2b})$$

Now,

$$E[\xi_T(t)] = E[\psi(t) - \psi(t - T)] = 0$$

$$E[\xi_T(t')] = 0$$

$$E(\sin 2p\theta_{1t}) = E\{\sin [2p[\omega_c t + \psi(t)] + 2p\phi_c]\}$$

For ϕ_c uniformly distributed on $[0, 2\pi]$, and $\psi(t)$ and ϕ_c independent,

$$\begin{aligned} E[\sin 2p\theta_{1t}] &= E\{\sin 2p[\omega_c t + \psi(t)]\} E[\cos 2p\phi_c] \\ &\quad + E\{\cos 2p[\omega_c t + \psi(t)]\} E[\sin 2p\phi_c] \\ &= 0 \end{aligned}$$

Similarly,

$$\left. \begin{aligned} E(\sin 2p\theta_{2t}) \\ E(\sin 2p\theta_{2t'}) \\ E(\sin 2p\theta_{1t'}) \end{aligned} \right\} = 0$$

Furthermore, by the independence of the random process $\psi(t)$ and the random variable ϕ_c

$$\begin{aligned} E[\xi_T(t) \sin 2p\theta_{1t'}] &= E\left[\psi(t) \sin \{2p[\omega_c t' + \psi(t')] + 2p\phi_c\}\right] \\ &\quad - E\left[\psi(t - T) \sin \{2p[\omega_c t' + \psi(t')] + 2p\phi_c\}\right] \\ &= 0 \end{aligned}$$

which further results in

$$\left. \begin{aligned} & E\{\xi_T(t') [\sin 2p\theta_{1t} - \sin 2p\theta_{2t}]\} \\ & E\{\xi_T(t) [\sin 2p\theta_{1t'} - \sin 2p\theta_{2t'}]\} \end{aligned} \right\} = 0$$

Therefore,

$$\begin{aligned} R_Z(t, t') = & \left(\frac{4}{\pi} \right)^2 \left\{ \omega_c^2 T^2 + R_{\xi_T}(t, t') \right. \\ & + E \left[\sum_{p=1}^{\infty} \sum_{q=1}^{\infty} \frac{(-1)^{p+q}}{pq} (\sin 2p\theta_{1t} - \sin 2p\theta_{2t}) \right. \\ & \left. \left. \cdot (\sin 2q\theta_{1t'} - \sin 2q\theta_{2t'}) \right] \right\} \quad (B.3) \end{aligned}$$

Further use may be made of the fact that

$$\begin{aligned} E[\sin 2p\theta_{1t} \sin 2q\theta_{1t'}] &= \frac{1}{2} E[\cos (2p\theta_{1t} - 2q\theta_{1t'})] \\ &\quad - \frac{1}{2} E[\cos (2p\theta_{1t} + 2q\theta_{1t'})] \\ &= \frac{1}{2} E \left[\cos (2p\omega_c t + 2p\phi_c + 2p\psi(t)) \right. \\ &\quad \left. - [2q\omega_c t' + 2q\phi_c + 2q\psi(t')] \right] \\ &\quad - \frac{1}{2} E \{ \cos [2p\omega_c t + 2p\phi_c + 2p\psi(t) \\ &\quad \quad + 2q\omega_c t' + 2q\phi_c + 2q\psi(t')] \} \\ &= 0 \quad \text{if } p \neq q \end{aligned}$$

Also, for all integral value of p and $q \in (1, \infty)$

$$\frac{1}{2} E[\cos (2p\theta_{1t} + 2q\theta_{1t'})] = 0$$

The other terms can be evaluated in a similar manner. Hence,

$$R_z(t, t') = \left(\frac{4}{\pi}\right)^2 \left[\omega_c^2 T^2 + R_{\xi_T}(t, t') + E \left\{ \frac{1}{2} \sum_{p=1}^{\infty} [\cos 2p(\theta_{1t} - \theta_{1t'}) \right. \right. \\ \left. \left. + \cos 2p(\theta_{2t} - \theta_{2t'}) - \cos 2p(\theta_{2t} - \theta_{1t'}) - \cos 2p(\theta_{1t} - \theta_{2t'})] \right\} \right] \quad (B.4)$$

Define $t - t' = \tau$, that is, $t' = t - \tau$

$$\begin{aligned} E[\cos 2p(\theta_{1t} - \theta_{1t'})] &= E\{\cos 2p[\omega_c t + \psi(t) - \omega_c t' - \psi(t')]\} \\ &= E\left[\cos 2p\{\omega_c \tau + [\psi(t) - \psi(t - \tau)]\}\right] \\ &= E\{\cos 2p[\omega_c \tau + \xi_{\tau}(t)]\} \quad \text{by definition of } \xi_{\tau}(t) \\ &= E\left[\operatorname{Re} \exp \{i[2p\omega_c \tau + 2p\xi_{\tau}(t)]\}\right] \\ &= E\left[\operatorname{Re} \{\exp [i2p\omega_c \tau] \exp [i2p\xi_{\tau}(t)]\}\right] \\ &= \operatorname{Re} \{\exp (i2p\omega_c \tau) E[\exp (i2p\xi_{\tau}(t))]\} \quad (B.5) \end{aligned}$$

By definition of the characteristic function,

$$\phi_{2p\xi_{\tau}}(f) = E[\exp (i\omega 2p\xi_{\tau})]$$

Here it is assumed that $\psi(t)$ is stationary. Therefore $\xi_\tau(t) = \psi(t) - \psi(t - \tau)$ is also stationary and depends on τ only. Hence

$$\phi_{2p\xi_\tau}\left(\frac{1}{2\pi}\right) = E\{\exp [i2p\xi_\tau(t)]\} \quad (B.6)$$

where $\phi_{2p\xi_\tau}$ is the characteristic function of $2p\xi_\tau$. Therefore

$$E[\cos 2p(\theta_{1t} - \theta_{1t'})] = \operatorname{Re}\left\{\exp (i2p\omega_c \tau) \phi_{2p\xi_\tau}\left(\frac{1}{2\pi}\right)\right\} \quad (B.7)$$

Similarly,

$$E[\cos 2p(\theta_{2t} - \theta_{2t'})] = E[\cos \{2p\omega_c \tau + 2p\xi_\tau(t - T)\}] \quad (B.8a)$$

$$E[\cos 2p(\theta_{2t} - \theta_{1t'})] = E[\cos \{2p\omega_c (\tau - T) + 2p\xi_{\tau-T}(t - T)\}] \quad (B.8b)$$

$$E[\cos 2p(\theta_{1t} - \theta_{2t'})] = E[\cos \{2p\omega_c (\tau + T) + 2p\xi_{\tau+T}(t)\}] \quad (B.8c)$$

This technique allows computation of any specific term, given $\psi(t)$.

To proceed further a specific case is assumed where $\psi(t)$ is a random process:

$$\psi(t) = \beta \sin (\omega_m t + \phi_m) \quad (B.9)$$

where ϕ_m is a random variable with

$$p(\phi_m) = \begin{cases} \frac{1}{2\pi} \in [0, 2\pi] \\ 0 & \text{elsewhere} \end{cases}$$

The following expansions by Jacobi and Anger are useful [Watson, Ref. 11, p. 22]:

$$\cos (z \sin \theta) = J_0(z) + 2 \sum_{n=1}^{\infty} J_{2n}(z) \cos 2n\theta \quad (\text{B.10a})$$

$$\sin (z \sin \theta) = 2 \sum_{n=0}^{\infty} J_{2n+1}(z) \sin (2n + 1)\theta \quad (\text{B.10b})$$

$$\cos (z \cos \theta) = J_0(z) + 2 \sum_{n=1}^{\infty} (-1)^n J_{2n}(z) \cos 2n\theta \quad (\text{B.10c})$$

$$\sin (z \cos \theta) = 2 \sum_{n=0}^{\infty} (-1)^n J_{2n+1}(z) \sin (2n + 1)\theta \quad (\text{B.10d})$$

Also, extensive use is made of the fact that for θ uniformly distributed on $(0, 2\pi)$ and with $f(t)$ and θ independent [Davenport and Root, Ref. 7]

$$E\{\cos [f(t) + \theta]\} = 0$$

Hence,

$$\begin{aligned} E[\cos 2p(\theta_{1t} - \theta_{1t},)] &= E\left[\cos \{2p[\omega_c \tau + \xi_\tau(t)]\}\right] \\ &= \cos 2p\omega_c \tau E[\cos 2p\xi_\tau(t)] \\ &\quad - \sin 2p\omega_c \tau E[\sin 2p\xi_\tau(t)] \end{aligned} \quad (\text{B.11})$$

Now,

$$\begin{aligned}
 E[\cos 2p\xi_\tau(t)] &= E\{\cos [2p\beta \sin (\omega_m t + \phi_m) - 2p\beta \sin (\omega_m t + \phi_m - \omega_m \tau)]\} \\
 &= E\{\cos [2p\beta \sin (\omega_m t + \phi_m)] \cos [2p\beta \sin (\omega_m t + \phi_m - \omega_m \tau)]\} \\
 &\quad + E\{\sin [2p\beta \sin (\omega_m t + \phi_m)] \\
 &\quad \cdot \sin [2p\beta \sin (\omega_m t + \phi_m - \omega_m \tau)]\}
 \end{aligned}$$

Using the above expansions and taking expectations,

$$\begin{aligned}
 E[\cos 2p\xi_\tau(t)] &= E\left\{J_0(2p\beta) + 2 \sum_{n=1}^{\infty} J_{2n}(2p\beta) \cos 2n(\omega_m t + \phi_m)\right. \\
 &\quad \cdot \left[J_0(2p\beta) + 2 \sum_{m=1}^{\infty} J_{2m}(2p\beta) \right. \\
 &\quad \cdot \left. \cos 2m(\omega_m t + \phi_m - \omega_m \tau)\right]\left. \right\} \\
 &\quad + E\left\{2 \sum_{n=0}^{\infty} J_{2n+1}(2p\beta) \sin (2n + 1)(\omega_m t + \phi_m)\right. \\
 &\quad \cdot \left[2 \sum_{m=0}^{\infty} J_{2m+1}(2p\beta) \sin (2m + 1)(\omega_m t + \phi_m - \omega_m \tau)\right]\left. \right\}
 \end{aligned}$$

$$\begin{aligned}
&= J_0^2(2p\beta) + 0 + 0 + 4 \sum_{n=1}^{\infty} \frac{J_{2n}^2(2p\beta)}{2} \cos 2n\omega_m \tau \\
&\quad + 4 \sum_{m=0}^{\infty} \frac{J_{2m+1}^2(2p\beta)}{2} \cos (2m+1)(\omega_m \tau) \\
&= J_0^2(2p\beta) + 2 \sum_{n=1}^{\infty} J_n^2(2p\beta) \cos n\omega_m \tau \\
&= \sum_{n=-\infty}^{\infty} J_n^2(2p\beta) \cos n\omega_m \tau \tag{B.12}
\end{aligned}$$

Also,

$$\begin{aligned}
E[\sin 2p\xi_\tau(t)] &= E\{\sin [2p\beta \sin (\omega_m t + \phi_m) - 2p\beta \sin (\omega_m t + \phi_m - \omega_m \tau)]\} \\
&= E\{\sin [2p\beta \sin (\omega_m t + \phi_m)] \cos [2p\beta \sin (\omega_m t + \phi_m - \omega_m \tau)]\} \\
&\quad - E\{\cos [2p\beta \sin (\omega_m t + \phi_m)] \sin [2p\beta \sin (\omega_m t + \phi_m - \omega_m \tau)]\} \\
&= E\left\{ \left[2 \sum_{n=0}^{\infty} J_{2n+1}(2p\beta) \sin (2n+1)(\omega_m t + \phi_m) \right] \right. \\
&\quad \left. \cdot \left[J_0(2p\beta) + 2 \sum_{m=1}^{\infty} J_{2m}(2p\beta) \cos 2n(\omega_m t + \phi_m - \omega_m \tau) \right] \right\}
\end{aligned}$$

$$\begin{aligned}
& - E \left\{ \left[J_0(2p\beta) + 2 \sum_{m=1}^{\infty} J_{2m}(2p\beta) \cos 2n(\omega_m t + \phi_m) \right] \right. \\
& \quad \cdot \left. \left[2 \sum_{n=0}^{\infty} J_{2n+1}(2p\beta) \sin (2n+1)(\omega_m t + \phi_m - \omega_m \tau) \right] \right\} \\
& = 0
\end{aligned} \tag{E.13}$$

Substituting (B.12) and (B.13) in (B.11) gives

$$\begin{aligned}
E \cos \{2p[\omega_c \tau + \xi_{\tau}(t)]\} &= \sum_{n=-\infty}^{\infty} J_n^2(2p\beta) \cos n\omega_m \tau \cos 2p\omega_c \tau \\
&= \sum_{n=-\infty}^{\infty} J_n^2(2p\beta) \cos (2p\omega_c + n\omega_m)\tau
\end{aligned} \tag{B.14}$$

It is obvious that

$$\begin{aligned}
E[\cos 2p(\theta_{2t} - \theta_{2t'})] &= E\{\cos [2p\omega_c \tau + 2p\xi_{\tau}(t - T)]\} \\
&= E\{\cos [2p\omega_c \tau + 2p\xi_{\tau}(t)]\} \\
&= \sum_{n=-\infty}^{\infty} J_n^2(2p\beta) \cos (2p\omega_c + n\omega_m)\tau
\end{aligned} \tag{B.15}$$

$$\begin{aligned}
E[\cos 2p(\theta_{2t} - \theta_{1t},)] &= E\{\cos [2p\omega_c(\tau - T) + 2p\xi_{\tau-T}(t - T)]\} \\
&= \cos 2p\omega_c(\tau - T) E[\cos 2p\xi_{\tau-T}(t - T)] \\
&\quad - \sin 2p\omega_c(\tau - T) E[\sin 2p\xi_{\tau-T}(t - T)]
\end{aligned}$$

It is evident from the above that

$$E[\cos 2p(\theta_{2t} - \theta_{1t},)] = \sum_{n=-\infty}^{\infty} J_n^2(2p\beta) \cos (2p\omega_c + n\omega_m)(\tau - T) \quad (B.16)$$

Similarly, it can be shown that

$$E[\cos 2p(\theta_{1t} - \theta_{2t},)] = \sum_{n=-\infty}^{\infty} J_n^2(2p\beta) \cos (2p\omega_c + n\omega_m)(\tau + T) \quad (B.17)$$

Hence

$$\begin{aligned}
&E\{\cos 2p(\theta_{1t} - \theta_{1t},) + \cos 2p(\theta_{2t} - \theta_{2t},) - \cos 2p(\theta_{2t} - \theta_{1t},) \\
&\quad - \cos 2p(\theta_{1t} - \theta_{2t},)\} = \sum_{n=-\infty}^{\infty} J_n^2(2p\beta) [2 \cos (2p\omega_c + n\omega_m)\tau \\
&\quad - \cos (2p\omega_c + n\omega_m)(\tau - T) - \cos (2p\omega_c + n\omega_m)(\tau + T)] \\
&= 2 \sum_{n=-\infty}^{\infty} J_n^2(2p\beta) [1 - \cos (2p\omega_c + n\omega_m)T] \\
&\quad \cdot \cos (2p\omega_c + n\omega_m)\tau
\end{aligned}$$

$$= 2 \sum_{n=-\infty}^{\infty} J_n^2(2p\beta) \sin^2 \left(2p\omega_c + n\omega_m \right) \frac{T}{2} \\ \cdot \cos \left(2p\omega_c + n\omega_m \right) \tau$$

Also,

$$R_{\xi_T}(t, t') = E\{\xi_T(t) \xi_T(t')\} \\ = E\{[\psi(t) - \psi(t - T)] [\psi(t') - \psi(t' - T)]\} \\ = E[\psi(t) \psi(t')] - E[\psi(t) \psi(t' - T)] \\ - E[\psi(t - T) \psi(t')] + E[\psi(t - T) \psi(t' - T)] \\ = 2R_\psi(\tau) - [R_\psi(\tau + T) + R_\psi(\tau - T)]$$

Taking the Fourier transform gives

$$S_{\xi_T}(f) = 2S_\psi(f) - [e^{i\omega T} + e^{-i\omega T}] S_\psi(f) \\ = 2S_\psi(f) [1 - \cos \omega T] \\ = 4S_\psi(f) \sin^2 \frac{\omega T}{2} \quad (B.18)$$

Hence,

$$R_z(t, t') = R_z(\tau) = \left(\frac{4}{\pi}\right)^2 \left\{ \omega_c^2 T^2 + R_{\xi_T}(\tau) + \sum_{p=1}^{\infty} \sum_{n=-\infty}^{\infty} \frac{2J_n^2(2p\beta)}{p^2} \right. \\ \left. \cdot \sin^2 \left[\left(2p\omega_c + n\omega_m \right) \frac{T}{2} \right] \cos \left(2p\omega_c + n\omega_m \right) \tau \right\}$$

Since $\psi(t) = \beta \sin(\omega_m t + \phi_m)$, it can be concluded that

$$\begin{aligned} R_\psi(\tau) &= E[\beta \sin(\omega_m t + \phi_m) \beta \sin(\omega_m t + \phi_m - \omega_m \tau)] \\ &= \frac{\beta^2}{2} \cos \omega_m \tau \end{aligned}$$

which implies

$$S_\psi(f) = \frac{\beta^2}{2} \frac{\delta(f - f_m) + \delta(f + f_m)}{2}$$

Hence in this case, autocorrelation of the output process,

$$\begin{aligned} R_z(t, t') = R_z(\tau) &= \left(\frac{4}{\pi}\right)^2 \left[\omega_c^2 T^2 + 2R_\psi(\tau) - [R_\psi(\tau + T) + R_\psi(\tau - T)] \right. \\ &\quad + \sum_{p=1}^{\infty} \sum_{n=-\infty}^{\infty} \frac{2J_n^2(2p\beta) \sin^2(2p\omega_c + n\omega_m) \frac{T}{2}}{p^2} \\ &\quad \left. \cdot \cos(2p\omega_c + n\omega_m)\tau \right] \end{aligned} \quad (B.19)$$

when the input random process $\psi(t) = \beta \sin(\omega_m t + \phi_m)$.

Eq. (B.19) is the result used extensively in Chapter III of this report.

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